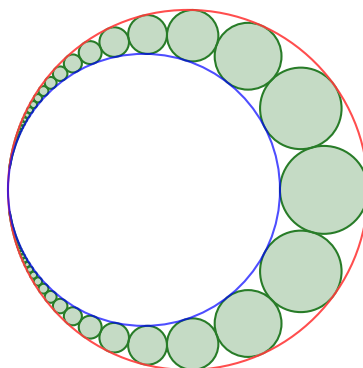


In this lab exercise you'll construct a *Pappus Chain*, a beautiful geometric object in which a ring of circles is contained inside of two others; see the figure below. The outer and inner circles are tangent to each other at one point. Each circle in the ring is tangent to both the inner and outer circle, as well as its two immediate neighbors.



It's possible to graph the Pappus Chain by calculating the center and the radius of each circle, but it's very messy. Instead, you'll construct a vertical column of circles, which is simpler, and use circle inversion to transform them into the Pappus Chain.

#### GEOGEBRA CONSTRUCTION – FIRST DRAFT

This isn't the construction you'll actually turn in, but it will show you how the Pappas Chain is created with circle inversions. Then you'll go back and create a more complicated version for your actual lab.

- (1) Open a new GeoGebra window. During the construction you should keep the axes and the grid on.
- (2) Create a circle named `mirror` centered at  $(-1, 0)$  with radius 2. You can do this with a circle tool, but it's easiest to type `mirror: (x+1)^2+y^2=4` in the Input field. We'll use this circle to create our image, but it won't be part of the Pappas Chain. You might want to change its style (make it dashed, change its color, etc.) to remind yourself of this.
- (3) Create the line  $x = 1$ , either with the graphical tools or by typing `x=1` in the Input field. Reflect the line across the mirror by choosing the *Reflect Across Circle* tool, or by typing `Reflect(f, mirror)` in the Input field, replacing `f` with the name of your line, as needed. The image of the line should be the unit circle, which will be the outer circle of our Pappas Chain.

*An aside: make sure you understand why the image is the unit circle. The line  $x = 1$  includes the point  $\infty$ , which is reflected to the center of the mirror, which is  $(-1, 0)$ . Meanwhile, the point  $(1, 0)$  is on the mirror, so it's fixed. The reflection of the rest of the line fills in the remaining points on the circle.*

- (4) Create the line  $x = 2$  and reflect it across the mirror. That gives you the inner circle in our chain. Because  $x = 2$  includes  $\infty$ , its reflection also goes through  $(-1, 0)$ , the center of the mirror, which is where the inner and outer circles are tangent to each other.
- (5) It's hard to find the equations of the circles in the Pappas Chain, but we're going to create a vertical column of circles between our two vertical lines, which is much easier. Here's where the magic

happens! GeoGebra includes a **Sequence** command which allows us to create a whole bunch of objects at once. Type the following in the Input field:

```
circles = Sequence( Circle( (3/2, k), 1/2 ), k, -50, 50)
```

(Make sure to hit the return key afterwards! If your version of GeoGebra creates a fraction when you type  $3/2$  or  $1/2$  you might have to press the right arrow to continue typing the rest of the line.) This command tells GeoGebra to create a circle centered at  $(3/2, k)$  with radius  $1/2$  for every  $k \in \{-50, -49, \dots, 49, 50\}$ . (Technically we want a column of infinitely many circles, but 101 circles will suffice for now!)

- (6) Type `pappus = Reflect(circles, mirror)` in the Input field to reflect the circles across the mirror to create your Pappus chain. Cool!

*Spend a minute thinking about why that last command created the Pappus chain. None of the circles in the vertical stack contain the center of the mirror. Therefore they're sent to other circles which do not contain the center of the mirror. Circle inversion preserves intersections, and each circle touches both lines  $x = 1$  and  $x = 2$  and two other circles. Therefore the stack of circles is sent to a chain of circles which are tangent to the images of those lines.*

#### GEOGEBRA CONSTRUCTION – FINAL VERSION

Now let's create a dynamic (i.e. adjustable) version of the Pappus Chain. We want to be able to adjust the size of the inner circle in the Pappus Chain, and have everything else update automatically. That means we need to adjust one of the vertical lines and the vertical stack of circles.

- (1) Starting from scratch, do the first three steps of the “First Draft” construction.
- (2) Select the *Point* tool and click on the  $x$ -axis between 0 and 1. This should create a point  $A$  which can only move along the axis. If you can move the point off the axis, delete it and try again. Then reflect  $A$  across `mirror` to get a point  $A'$ .
- (3) Create a vertical line through  $A'$ . One way to do this is by typing `x = x(A')` in the Input field. The right hand side of that expression is GeoGebra-ese for “the  $x$ -coordinate of  $A'$ .” Reflect this line across `mirror`. The reflection should be a circle that goes through  $A$  (because  $A$  and  $A'$  are reflections of each other, after all) and  $(-1, 0)$  (because the line includes  $\infty$ , which is reflected to the center of `mirror`).
- (4) Before continuing: check your work to this point by sliding  $A$  back and forth along the axis, and making sure your second vertical line slides back and forth.
- (5) The radius of the circles in our vertical stack will be one half the distance between the two vertical lines. Type `r = Distance(f,g)/2` in the input line to define the radius. (Replace `f` and `g` with the names of your vertical lines if necessary.) Notice that its value changes as you slide  $A$  around.
- (6) Now it's time to create the circles again. We have to adjust our **Sequence** command a bit. The  $x$ -coordinate of the centers is now  $1 + r$ , instead of  $1 + 1/2$ , and the vertical coordinates will be  $2rk$  instead of  $k$ . Type the following in the Input field:

```
circles = Sequence(Circle( (1+r, 2r k), r), k, -50, 50)
```

(Make sure to hit the return key afterwards!) This command tells GeoGebra to create circles centered at  $(1 + r, 2rk)$  for  $k \in \{-50, -49, \dots, 49, 50\}$ .

- (7) Type `pappus = Reflect(circles, mirror)` in the Input field to reflect the circles across the mirror to create your Pappus chain.

- (8) Check your construction by dragging the point  $A$  back and forth, and making sure the chain updates accordingly. Very cool!
- (9) Go back and clean up your diagram so that it looks like the picture at the top of this document, with the addition of point  $A$  that you can drag back and forth. None of the vertical lines, vertical stack of circles, axes, grid, mirror, or labels should be displayed. (You don't need to adjust the colors of the circles, but you're welcome to if you want!) You should also adjust your window and the zoom level appropriately; your construction shouldn't be a small circle in one corner of the window.

To receive credit for this assignment, save your file as `lastname-5335-1ab5.ggb` and email it to me as an attachment by the beginning of class on Wednesday, 11/20/19. I'll use the following grading rubric out of 5 points:

**2 Points:** I can open your file, and see a static Pappus Chain, as described in the "First Draft" section.

**4 Points:** I can open your file and see a dynamic construction, as described in the "Final Version" section, but you have left labels, axes, grids or other unnecessary objects displayed.

**5 Points:** I can open your file, the construction is correct, and it's been cleaned up appropriately.