The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

3.7.14: There are infinitely many answers for this question. If you didn't come up with anything, I'd suggest drawing a picture with these four points: $A = (1, 0)$, $B = (-1, 0)$, $P = (0, 1)$ and $Q = (0, -1)$.

3.7.15: Let A and B be two linearly independent vectors such that $||A|| = ||B||$. Drawn with their tails at the origin, the heads of both vectors will be on the some circle through the origin. Let l be the line through the origin which contains the angle bisector of $\angle AOB$. Then a reflection across l is an isometry which takes A to B and vice versa. (Draw a picture of this!!) *Note that this isometry sends* $(0, 0)$ *to* $(0, 0)$.

To solve Problem 46 from chapter 1, let $\mathcal{U}$ be an isometry which takes U to V and V to U. If $\alpha = 1$ then the problem is easy, so assume $\alpha \neq 1$. Then

$$\begin{aligned}
||U + \alpha V|| &= ||\mathcal{U}(U) + \mathcal{U}(\alpha V)|| && \text{(definition of isometry)}\\
&= ||V + \mathcal{U}\left(\alpha V + (1-\alpha)O\right)|| && \text{(clever addition of the vector } O = (0,0))\\
&= ||V + \alpha\mathcal{U}(V) + (1-\alpha)\mathcal{U}(O)|| && \text{(Lemma 4)}\\
&= ||V + \alpha U + O|| = ||\alpha U + V||
\end{aligned}$$

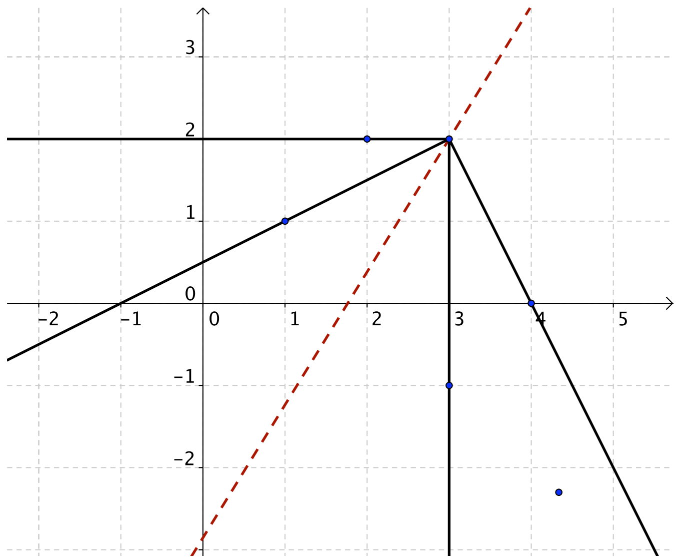
3.7.20: To me this problem is most difficult because of the notation, especially $\mathcal{U}$ and U. Also, Y is the *input* for the original isometry and X is the output. (Hence X is the input of $\mathcal{U}^{-1}$ and Y is the output of $\mathcal{U}^{-1}$.) We may assume U is a $2x2$ orthogonal matrix in the given formula; to avoid overuse of the letter, I will replace it with M:

$$X = \mathcal{U}(Y) = MY + P$$

To find the inverse, solve that matrix/vector equation for Y in terms of X:

$$\begin{aligned}
X &= MY + P\\
M^{-1}X &= M^{-1}MY + M^{-1}P = Y + M^{-1}P\\
M^{-1}X - M^{-1}P &= Y
\end{aligned}$$

3.7.28: Here's a picture with the relevant points and angles. There are two isometries that come to mind which would send the first angle to the second. The first would be a reflection across the dotted line. To my knowledge not many people did this, in part because the equation of that specific line of reflection is tricky to find and not particularly nice.



The other possibility is to rotate the plane $90° = \pi/2$ counterclockwise about the point $(3, 2)$. To write down a formula for this takes a bit of work, because the rotation matrix we know is only for rotations about the origin:

$$M = \begin{bmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

To use this matrix, you first need to move the point $(3, 2)$ to the origin, then do the rotation, and then move everything back to where it came from:

$$\begin{aligned} \mathcal{U}(X) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(X - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -y \\ x \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 - y \\ x - 1 \end{bmatrix} \end{aligned}$$

The way we constructed $\mathcal{U}$ it's clear that the horizontal ray rotates $90°$ onto the vertical ray, but it's also worth checking the other part of the angle. You can calculate that $\mathcal{U}(1, 1) = (4, 0)$ using the formula above.