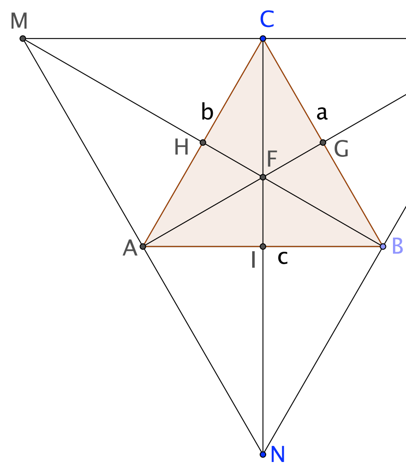
The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

4.11.51: From previous problems we know that, if a triangle is equilateral, then the centroid, incenter and orthocenter are all the same; because the altitudes are also the perpendicular bisectors of the sides, we can toss the circumcenter into that list as well. In this problem we prove the Fermat Point belongs in that list, too. The following picture shows the construction of the Fermat Point F for a triangle $\triangle ABC$, as described in section 4.9.



Suppose $\triangle ABC$ is equilateral. Then all six of the interior triangles inside $\triangle ABC$ are congruent (this follows quickly from, say problem 4.11.26 on the previous assignment). In particular, G, H, and I must be the midpoints of the sides. But then the segments $\overline{AL}$, $\overline{BM}$ and $\overline{CN}$ which are used to define the Fermat Point coincide with the medians, proving that F is also the centroid.

Conversely, suppose we know that the Fermat Point F happens to be the centroid, and we wish to prove that $\triangle ABC$ is equilateral. We can still use the diagram above, except we can't assume $a = b = c$ or anything equivalent – that's what we're trying to prove! We know that the exterior triangles are equilateral, but we don't yet know that they're the same size.

Here's what we do know. Since F is the centroid, $\overline{AG}$ is a median and $|\overline{GC}| = |\overline{GB}|$. Hence (by SSS in Chapter 3) we know $\triangle LGC \cong \triangle LGB$. But then $|\angle LGC| = |\angle LGB|$ and these angles add to form a straight angle, meaning each of them is $\pi/2$. Hence $|\angle FGC| = |\angle FGB| = \pi/2$ as well, and $\overline{AG}$ is not only a median but also an altitude and perpendicular bisector! By symmetry the same holds true for $\overline{BH}$ and $\overline{CI}$. So now F is not only the Fermat Point and the centroid, but also the orthocenter and circumcenter!.

Because F is the circumcenter, $|\overline{FA}| = |\overline{FB}| = |\overline{FC}|$. Also, because F is the Fermat Point, $|\angle AFB| = |\angle BFC| = |\angle CFA| = 2\pi/3$. Hence (by SAS in Chapter 3),

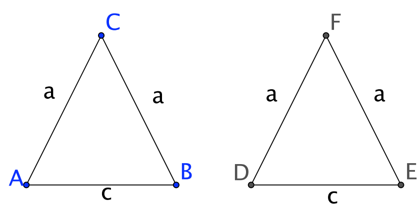
$$\triangle ABF \cong \triangle BCF \cong \triangle CAF$$

from which it quickly follows that $a = b = c$, so $\triangle ABC$ is equilateral.

5.3.1: Differentiate using the power rule and chain rule, etc.:

$$\begin{aligned}\frac{d}{dx}\sin x = \frac{d}{dx}\sqrt{1-\cos^2 x} &= \left(1-\cos^2 x\right)^{1/2} \\ &= \frac{1}{2}(1-cos^2 x)^{-1/2}\cdot(-2(\cos x)(\cos x)') \\ &= \frac{1}{2}\cdot\frac{1}{\sin x}\cdot(-2(\cos x)(-\sin x)) = \cos x\end{aligned}$$

5.3.8: This is one of those sneaky problems where there's very little to do—so little that, unfortunately, it can be hard to figure out what to write! With that in mind I'll explain this one in excruciating detail. First, suppose $\triangle ABC$ is isoceles, as shown in the following picture.



Using a translation I can create a a congruent "copy" $\triangle DEF \cong \triangle ABC$. That means six things:

$$\begin{array}{ll} \overline{AB}\cong\overline{DE} & \angle ABC \cong \angle DEF \\ \overline{BC}\cong\overline{EF} & \angle BCA \cong \angle EFD \\ \overline{CA}\cong\overline{FD} & \angle CAB \cong \angle FDE \end{array}$$

However, because $\triangle ABC$ is isoceles, I know $\overline{AC}\cong\overline{BC}\cong\overline{DF}\cong EF$; they all have length a as shown in the picture. Hence I can write a *second* congruence using SSS from Chapter 3: $\triangle ABC \cong \triangle EDF$, which means the following six things:

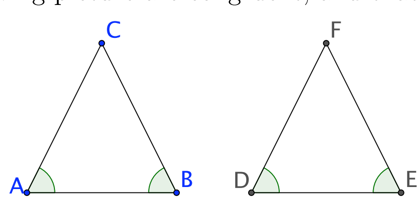
$$\begin{array}{ll} \overline{AB}\cong\overline{ED} & \angle ABC \cong \angle EDF \\ \overline{BC}\cong\overline{DF} & \angle BCA \cong \angle DFE \\ \overline{CA}\cong\overline{FE} & \angle CAB \cong \angle FED \end{array}$$

Compare those two tables until you see and understand the differences!. We really only need two pieces of information; one from the first table and one from the second:

$$\angle CAB \cong \angle FDE \cong \angle ABC$$

which proves that the two vertex angles which are opposite the equal sides are also congruent.

The other direction is very similar, but requires a slightly different picture. Now we assume that all of the marked angles in the following picture are congruent, and that $|\overline{AB}| = |\overline{DE}|$.



Similar to the first part of the problem, I can use the ASA congruence criterion to say that $\triangle ABC \cong \triangle DEF$ as well as $\triangle ABC \cong \triangle EDF$. If you're having trouble understanding this problem, you should write out the six things each congruence statement tells you. All we need is that $\overline{AC}\cong\overline{DF}$ (via the first congruence) and $\overline{DF} \cong \overline{BC}$ (via the second congruence). Hence $\overline{AC} \cong \overline{BC}$ and $\triangle ABC$ is isoceles.

5.3.9: This problem just requires repeated application of Proposition 15, which we proved in the previous exercise. Suppose $a = b = c$. Then $a = b$ implies $\alpha = \beta$, $b = c$ implies $\beta = \gamma$, and hence all three angles are equal. (Draw a picture and label all three sides and all three angles!) In the other direction, if $\alpha = \beta = \gamma$, then $\alpha = \beta$ implies $a = b$, $\beta = \gamma$ implies $b = c$, and thus $a = b = c$.

5.3.12: (ii) Use Theorem 3 and the fact that $\cos -\phi = \cos\phi$, while $\sin -\phi = -\sin\phi$.

(iii) Use part (i) and the above facts about $\cos -\theta$, $\sin -\theta$:

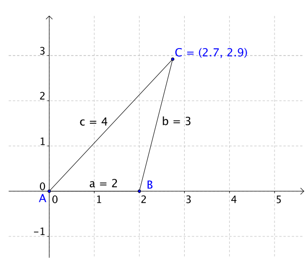
$$\begin{aligned} \cos 2\phi &= \cos(\phi - (-\phi)) = \cos\phi\cos\phi - \sin\phi\sin\phi \\ &= \cos^2\phi - \sin^2\phi \\ &= \cos^2\phi - (1 - \cos^2\phi) \\ &= 2cos^2\phi - 1 \end{aligned}$$

Rearranging gives

$$\begin{aligned} \cos^2\phi &= \frac{1}{2}(1 + \cos(2\phi)) \\ \cos\phi &= \pm\sqrt{(1 + \cos(2\phi))/2} \end{aligned}$$

The substitution $\phi = \theta/2$ finishes this off; the sign in front of the square root depends on the value of ϕ (or θ, if you prefer).

5.3.18: (i) Yes, such a triangle exists, because a, b, c satisfy the inequalities in Proposition 9. Here's a picture of such a triangle (where GeoGebra has rounded the coordinates):



(iii) No such triangle exists, by Proposition 9, because 5 is not less than $2 + 2$.

5.3.23: If $\tan x = \frac{\sin x}{\cos x}$, then the domain is all x for which $\cos x \neq 0$, which is the set of all real numbers except $x = \pi/2 + k\pi, k \in \mathbb{Z}$. We can compute the derivative directly using the quotient rule:

$$\begin{aligned} \frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} &= \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x} \\ &= \frac{cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$