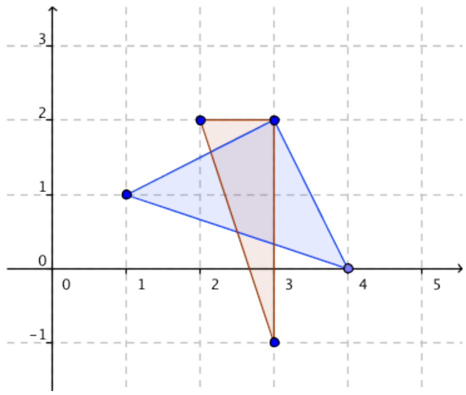
The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

3.6.29: This type of problem will be discussed in much more detail in Chapter 7, so I will just describe the basic construction here; even so, it's long and somewhat confusing, so if you'd like more explanation I'll walk you through it in person. To start with, I've drawn a picture of the six points. Notice that I will not be able to find an isometry which maps the first three points onto the second; that would mean the two *triangles* in the picture are congruent, which is visibly false.



However, the two blue segments emanating from $(3,2)$ form a right angle, as do the two orange segments. So if I could rotate the entire plane about $(3,2)$ just enough to rotate the blue right angle onto the orange right angle, I'll be done.

How much of a rotation do I need? Let $\theta = |\angle(1,1)(3,2)(2,2)|$. I don't know what θ is, although by drawing in a right triangle–say, from $(1,1)$ to $(3,2)$ to $(1,2)$–I can see that

$$\cos\theta = 2/\sqrt{5} \qquad \text{and} \qquad \sin\theta = 1/\sqrt{5}$$

Because I don't know how to rotate at any point except the origin, I move everything everything there:

$$X \to X - \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Now I use the rotation matrix which rotates the plane by an angle of θ:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

EXCEPT... R_θ rotates in a counter-clockwise direction about the origin, and I want to rotate clockwise. So I replace θ with $-\theta$ to change the direction:

$$R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

So after moving to the origin and rotating, I have

$$X \to R_{-\theta}\left(X - \begin{bmatrix}3\\2\end{bmatrix}\right)$$

To finish, I move everything back to where I started:

$$\begin{aligned}
\mathcal{U}(X) &= R_{-\theta}\left(X - \begin{bmatrix}3\\2\end{bmatrix}\right) + \begin{bmatrix}3\\2\end{bmatrix}\\
&= \begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\ -1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix}\left(X - \begin{bmatrix}3\\2\end{bmatrix}\right) + \begin{bmatrix}3\\2\end{bmatrix}\\
&= \begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\ -1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix} X + \begin{bmatrix}3+8/\sqrt{5}\\ 2+1/\sqrt{5}\end{bmatrix}
\end{aligned}$$

You can plug in $X = \begin{bmatrix}4\\0\end{bmatrix}$ to verify that $\mathcal{U}$ sends $(4,0)$ to a point on the line $x = 3$; similarly, $\mathcal{U}(1,1)$ is on the line $y = 2$. So this $\mathcal{U}$ maps the blue right angle onto the one formed by the horizontal and vertical orange line segments.

4.11.9: If $\triangle ABC$ is equilateral, then $a = b = c$, so the incenter (by Theorem 4.17) is

$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c}\right)^{\triangle} = \left(\frac{a}{a+a+a}, \frac{a}{a+a+a}, \frac{a}{a+a+a}\right)^{\triangle} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle},$$

which is the centroid. Conversely, suppose the centroid and incenter are equal:

$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c}\right)^{\triangle} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle}$$

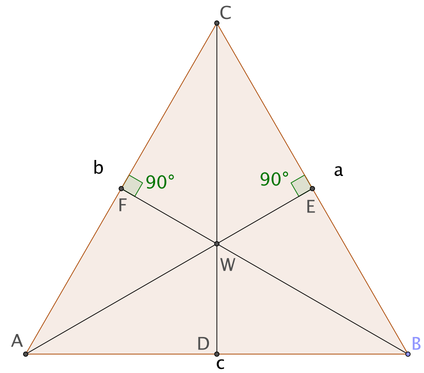
Equating the parts of the barycentric coordinates (and multiplying by 3) gives three equations:

$$\begin{aligned}
3a &= a+b+c\\
3b &= a+b+c\\
3c &= a+b+c
\end{aligned}$$

You can check that this system of equations is satifsfied if and only if $a = b = c$, so that the triangle is equilateral. (The fastest way to solve this system: notice that $3a$, $3b$ and $3c$ are all equal to each other...)

4.11.24: If $\triangle ABC$ is equilateral, then $a = b = c$, so you can use the barycentric coordinates of the incenter and orthocenter to show these are both (after much simplication) equal to $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle}$.

In the other direction, suppose the incenter and orthocenter are the same point; in the following picture I'll refer to it as W. First, note that the angle bisectors must equal the altitudes, which is why I only have one line emanating from each vertex. (Why is this the case? Consider the vertex C on top. The angle bisector and altitude each start at C and go through W by our assumption. Because they share two points, they must be contained in the same line!)



In particular, because $\overline{AE}$ is an angle bisector, we have

$$\angle BAE \cong \angle CAE$$

But $\overline{AE}$ is also an altitude, meaning

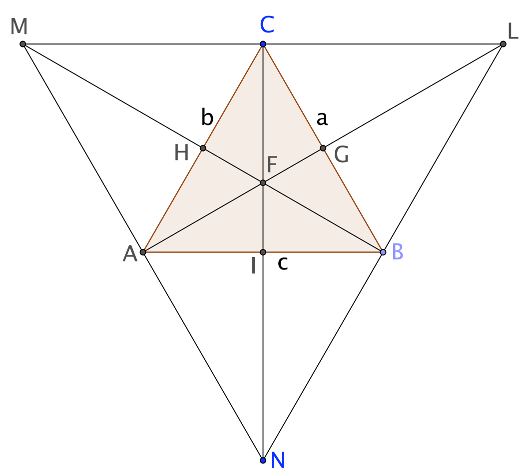
$$\angle AEB \cong \angle AEC$$

(They're both right angles.) Via email I told everybody you could use the ASA congruence theorem for this problem, so we apply it and conclude

$$\triangle AEC \cong \triangle AEB$$

Hence $\overline{AC} \cong \overline{AB}$ – that is, $b = c$. Now you could do the same work (starting at the vertex B instead of A) to show that $a = c$. So overall $a = b = c$ and $\triangle ABC$ is equilateral.

4.11.44: From previous problems we know that, if a triangle is equilateral, then the centroid, incenter and orthocenter are all the same; because the altitudes are also the perpendicular bisectors of the sides, we can toss the circumcenter into that list as well. In this problem we prove the Fermat Point belongs in that list, too. The following picture shows the construction of the Fermat Point F for a triangle $\triangle ABC$, as described in section 4.9.



Suppose we know that the Fermat Point F happens to be the centroid, and we wish to prove that $\triangle ABC$ is equilateral. We can use the diagram above, except we can't assume $a = b = c$ or anything equivalent – that's what we're trying to prove! We know that the exterior triangles are equilateral, but we don't yet know that they're the same size.

Here's what we do know. Since F is the centroid, $\overline{AG}$ is a median and $|\overline{GC}| = |\overline{GB}|$. Hence (by SSS in Chapter 3) we know $\triangle LGC \cong \triangle LGB$. But then $|\angle LGC| = |\angle LGB|$ and these angles add to form a straight angle, meaning each of them is $\pi/2$. Hence $|\angle FGC| = |\angle FGB| = \pi/2$ as well, and $\overline{AG}$ is not only a median but also an altitude and perpendicular bisector! By symmetry the same holds true for $\overline{BH}$ and $\overline{CI}$. So now F is not only the Fermat Point and the centroid, but also the orthocenter and circumcenter!

Because F is the circumcenter, $|\overline{FA}| = |\overline{FB}| = |\overline{FC}|$. Also, because F is the Fermat Point (which by Theorem 28 is also the Fermat minimizer) $|\angle AFB| = |\angle BFC| = |\angle CFA| = 2\pi/3$. Hence (by SAS in Chapter 3),

$$\triangle ABF \cong \triangle BCF \cong \triangle CAF$$

from which it quickly follows that $a = b = c$, so $\triangle ABC$ is equilateral.

Conversely, suppose $\triangle ABC$ is equilateral and let F be the centroid. By previous problems it's also the incenter and orthocenter, and $\overline{AG}$, $\overline{BH}$ and $\overline{CI}$ are medians, angle bisectors, and altitudes. (For this part of the problem, ignore L, M, N, and any of the segments which are outside of the original triangle.) Using ASA or other congruence theorems, you can quickly show that all six of the little triangles formed inside $\triangle ABC$ are congruent. That means each of the six angles surrounding the centroid F are congruent, so each one is $360°/6 = 60°$. But then

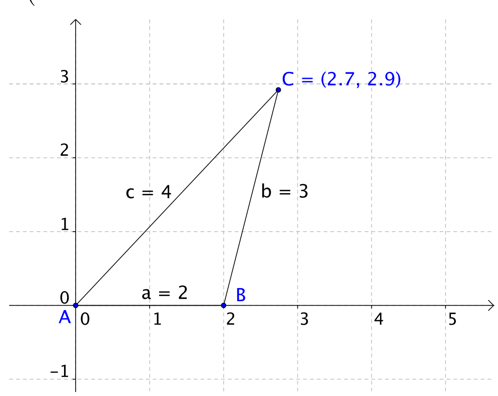
$$|\angle AFC| = |\angle CFB| = |\angle BFA| = 120° = 2\pi/3$$

But forming those angles of $120°$ proves that the centroid F is the Fermat *minimizer* and therefore, by Theorem 28, the Fermat Point.

5.3.1: Differentiate using the power rule and chain rule, etc.:

$$\begin{aligned}\frac{d}{dx}\sin x = \frac{d}{dx}\sqrt{1-\cos^2 x} &= \left(1-\cos^2 x\right)^{1/2} \\ &= \frac{1}{2}(1-cos^2x)^{-1/2}\cdot(-2(\cos x)(\cos x)') \\ &= \frac{1}{2}\cdot\frac{1}{\sin x}\cdot(-2(\cos x)(-\sin x)) = \cos x\end{aligned}$$

5.3.18: (i) Yes, such a triangle exists, because a, b, c satisfy the inequalities in Proposition 9. Here's a picture of such a triangle (where GeoGebra has rounded the coordinates):



(iii) No such triangle exists, by Proposition 9, because 5 is not less than $2 + 2$.

5.3.23: If $\tan x = \frac{\sin x}{\cos x}$, then the domain is all x for which $\cos x \neq 0$, which is the set of all real numbers except $x = \pi/2 + k\pi, k \in \mathbb{Z}$. We can compute the derivative directly using the quotient rule:

$$\begin{aligned}\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} &= \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x} \\ &= \frac{cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

For $x \in (-\pi/2, \pi/2)$, $\frac{d}{dx}\tan x = \sec^2 x$ is defined and positive, which means $\tan x$ is strictly increasing on that interval. As discussed in the solutions to Homework #2, that implies $\tan x$ is injective on that interval.