Chapters 1 and 2 - Flythrough

This will be a fast review/intro – You're responsible for reading Chapter 1 (and 2) and talking to me if there's something you don't follow.

(Think: big ideas/terms, like "line," not the intricacies of, say, proof of Proposition 1.30...)

Fundamentals / Vocabulary

$\forall$: for all, for every

$\exists$: there exists

iff : if and only iff, $\Longleftrightarrow$

We won't use sets in much depth. Mostly:

$\mathbb{R}$ = real #'s

$\mathbb{R}^2 = \{ (x,y) : x, y \in \mathbb{R} \} = \{ (x,y) \mid x, y \in \mathbb{R} \}$

and esp. subsets of those

Recall "Abstract" Function Notation

$f: A \to B$ A: domain, inputs

$x \mapsto f(x)$ B: codomain (range, image)

$x \rightsquigarrow f(x)$ set of possible outputs

Ex $f: \mathbb{R} \to \mathbb{R}$ or $\{x : x \geq 0\}$

$x \mapsto x^2$

$(f(x) = x^2)$

Def f is injective (or one-to-one, 1:1) if any two different inputs are sent to diff. outputs: $x \neq y \Rightarrow f(x) \neq f(y)$

f is surjective (onto) if every elt of codomain is actual output:

$\forall\, b \in B$, there exists $a \in A$ such that $f(a) = b$.

Functions Sheet

	Injective	Surjective
1.	✗	✓
2.	✓	✗
3.	✓	✗
4.	✓	✓
5.	✓	✗
6.	✓	✗
7.	✓	✗
8.	✗	✗
9.	✗	✗
10.	Not a Function	

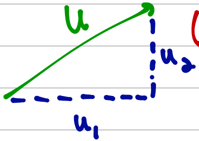
Vectors, Points and Lines

A (2D) vector is an ordered pair of real #'s, (a,b).

Common notations: $\langle a,b\rangle$, $\overrightarrow{(a,b)} = \vec{u} = (u_1, u_2)$

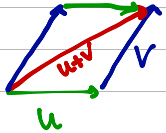
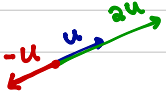
Our book: $U = (u_1, u_2)$, $X = (x_1, x_2)$

Graphically, U is an arrow:



(here $u_1, u_2 > 0$)

Virtually every vector concept has an algebraic def^n/ interpretation and a geometric/axiomatic one.

	alg	geo
Addition	$U+V=(u_1,u_2)+(v_1,v_2)$ $=(u_1+v_1,\ u_2+v_2)$	
(scalar) mult'n	$cU=c(u_1,u_2)$ $=(cu_1,\ cu_2)$	
Subtraction	$U-V=U+(-1)V$	
linearly dependent	$U=cV$ or $V=cU$ $\exists\ a,b$ such that $aU+bV=0,$ a,b not both 0.	U, V are $\parallel$

alg | geo

$u \cdot v$, $\langle u, v \rangle$

dot product:
scalar product,
inner product

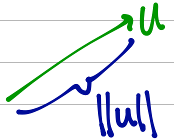
$$u \cdot v = (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2$$

?

$$u \cdot u = (u_1, u_2) \cdot (u_1, u_2) = u_1^2 + u_2^2$$

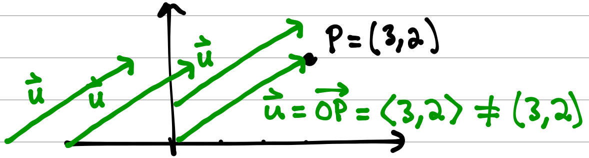
length/
magnitude

$$\|u\| = \sqrt{u \cdot u}$$



⚠ If you learned vectors from Stewart's book...

Stewart make huge distinctions between points and vectors:

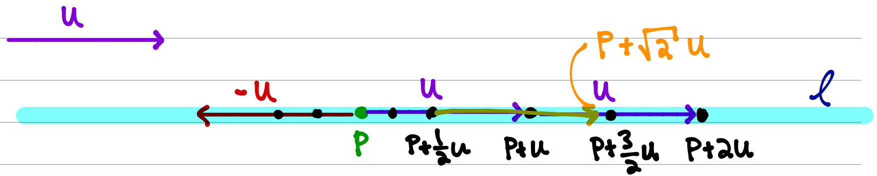


<u>We won't</u>. For us, vector and point are synonyms.

It's clear from context, and it makes life easier to do it this way. To wit:

Def Given a point P and non-zero vector U, the set

$\ell = \{P + sU : s \in \mathbb{R}\}$ is a line.



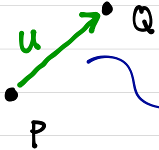
U is direction indicator (dir'n vector). Points on ℓ are incident with ℓ, and are collinear.

Ex $(-1,2)+s(3,-4)$ $\quad$ $(\frac{1}{2},0)$ is on line $(s=\frac{1}{2})$

$= (-1+3s,\ 2-4s)$

$(5,6)$ is not. $\left.\begin{array}{l} -1+3s=5 \\ 2-4s=6 \end{array}\right\}$ no sol'n

⚠ Important Example. What's u?



$$P + u = Q$$

$$u = Q - P$$

Warmup Problem (after break)

Recall: $U \cdot V = (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2$

Prove: For all $c \in \mathbb{R}$,

$$(cU) \cdot V = U \cdot (cV) = c(U \cdot V)$$

Pf: $(cU) \cdot V = (cu_1, cu_2) \cdot (v_1, v_2) = cu_1 v_1 + cu_2 v_2$

$$= u_1(cv_1) + u_2(cv_2)$$

$$= U \cdot (cV)$$ etc.

Warmup Problem (after break)

Recall: $U \cdot V = (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2$

Prove: The dot product is commutative: $U \cdot V = V \cdot U$

$$U \cdot V = u_1 v_1 + u_2 v_2 = v_1 u_1 + v_2 u_2 = V \cdot U$$

Prove: The dot product is distributive: $U \cdot (V + W) = U \cdot V + U \cdot W$

$$U \cdot (V + W) = (u_1, u_2) \cdot (v_1 + w_1, v_2 + w_2)$$

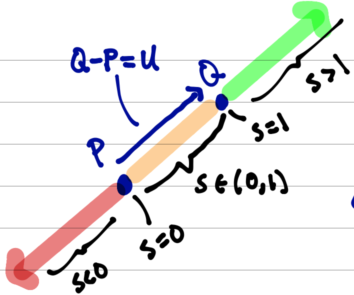
$$= u_1(v_1 + w_1) + u_2(v_2 + w_2)$$

$$= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2$$

$$= (u_1 v_1 + u_2 v_2) + (u_1 w_1 + u_2 w_2)$$

$$= U \cdot V + U \cdot W$$

$P + s(Q-P)$



$$\overline{PQ} = \overline{QP}$$
$$\{P + s(Q-P)\} \qquad \{Q + s(P-Q)\}$$

$$\overleftrightarrow{PQ} = \overleftrightarrow{QP}$$

$$\overrightarrow{PQ} \neq \overrightarrow{QP}$$

$$\|X\| = \sqrt{X \cdot X}$$

length of $\overline{PQ}$ is $\|Q-P\| = \|P-Q\|$ $\leftarrow$ $\|u\| = \|-u\|$

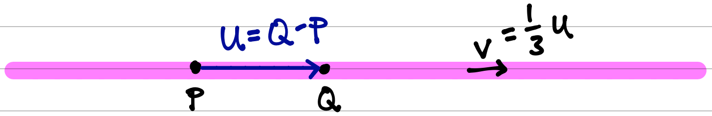
$$= \sqrt{\langle Q-P,\ Q-P\rangle}$$

$$= \sqrt{(Q-P)\cdot(Q-P)}$$

day 1

Does the def^n $\{P+sU\}$ cover everything we expect?

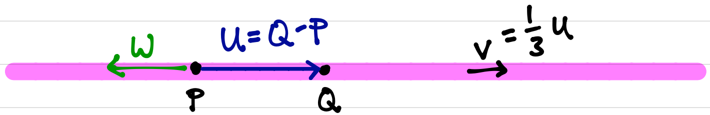
- Can get segments, rays using restricted values of s.
- Two points form a line? Yes (wksheet)
- Is that line unique?!



$$\{P+sU\} = \{Q+tV\}$$

Prop 1.4 two non-zero vectors are DI's of same line
iff they're scalar mult's of each other.

Let $P \neq Q$. Then $\exists$ unique line $\overleftrightarrow{PQ}$ incident with both, $u = Q - P$ is a DI of $\overleftrightarrow{PQ}$, and every DI of $\overleftrightarrow{PQ}$ is difference of two pts on the line.



Other Forms

Ex $(-1,2)+s(3,4) = (-1,2)+(3s,4s) = (\underbrace{-1+3s}_{x}, \underbrace{2+4s}_{y})$

$$\left.\begin{aligned} x &= 3s-1 \\ y &= -4s+2 \end{aligned}\right\} \Rightarrow \begin{aligned} s &= \tfrac{1}{3}(x+1) \\ s &= -\tfrac{1}{4}(y-2) \end{aligned}$$

$$-\frac{1}{4}(y-2) = \frac{1}{3}(x+1)$$

pt slope form $\longrightarrow$ $y-2 = -\frac{4}{3}(x+1)$

$$y = -\frac{4}{3}x - \frac{4}{3} + 2$$

slope intercept $\longrightarrow$ $y = \frac{-4}{3}x + \frac{2}{3}$

Def Two lines ℓ, m are parallel, $\ell \parallel m$, if their DI's are $\parallel$.

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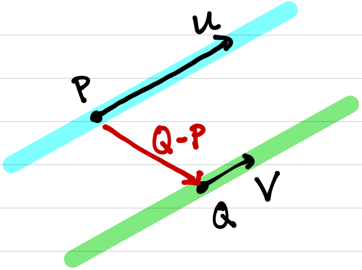
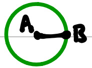
Prop 1.6 Lines $\ell = \{P + sU\}$, $m = \{Q + tV\}$



- $\cap$ in <u>one</u> pt if U, V linearly independent (<u>not</u> $\parallel$)
- empty $\cap$'n if $U \parallel V$ and $U \nparallel Q-P$
- same line if $U \parallel V$ <u>and</u> $U \parallel Q-P$

Quick Status Check: which of Euclid's Axioms work so far?

1. Given two pts, $\exists$ line containing them — yes
2. Lines can be extended indefinitely — yes
3. Given A, B, $\exists$ circle cent'd at A with radius $\overline{AB}$.



$r = \|B - A\|$
$C : \{X : \|X - A\| = r\}$

4. Right angles are all equal — ? (Chapter 3)
5. $\parallel$ postulate — Yes – HW

Perependicularity / Orthogonality

$U \cdot V = u_1 v_1 + u_2 v_2$

Def $U \perp V$ if $U \cdot V = 0$. Two lines are perpendicular if their DI's are $\perp$.

$\parallel$ and $\perp$ play important roles...

Corollary 1.11 If ℓ is a line and P is a point, $\exists$ exactly one line incident with P and $\parallel$ to ℓ.

Prop 1.15 If ℓ is a line and P is a point, $\exists$ exactly one line incident with P and $\perp$ to ℓ.

Corollary 1.16 (lemma) The set of vectors $\perp$ to $U=(u_1,u_2)\neq 0$ consists of all multiples of $(-u_2,u_1)$

Pf: First, we see that $(u_1,u_2)\cdot(-u_2,u_1) = -u_1u_2+u_2u_1=0.$

Now suppose $V\perp U$, so that $u_1v_1+u_2v_2=0$. We want to show $V=c(-u_2,u_1)$ for some c.

i.e. $v_1=c(-u_2)$ and $v_2=c(u_1)$

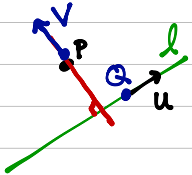
Assume $u_1\neq 0$. Then $v_1=\frac{-u_2v_2}{u_1}=\underbrace{\frac{v_2}{u_1}}_{c}(-u_2)$

and then $c(u_1)=\left(\frac{v_2}{u_1}\right)u_1=v_2$,

as desired.

(left for you to finish)

Prop 1.15 If ℓ is a line and P is a point, $\exists$ exactly one line incident with P and $\perp$ to ℓ.



Pf Suppose $\ell: Q + sU$, $U \neq 0$

Then $m: P + s\underbrace{(-u_2, u_1)}_{V \perp U \text{ by prev. slide.}}$ is $\perp \ell$.

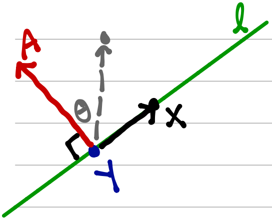
Suppose $\exists$ another such line $P + sW$, so $U \perp W$

By prev. slide, $W = cV$ for some c.

By Prop 1.6, they are same line.

Normal Form

Given line ℓ, choose $Y \in \ell$ and $A \perp \ell$ (i.e. $A \perp u$, u **any** DI of ℓ). Then



$$\ell = \{X : A \cdot (X - Y) = 0\}$$

$$\left(= \{X : A \perp (X - Y)\} \right)$$

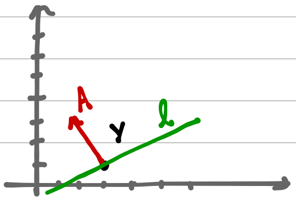
⚠ A, Y **constants** $X = (x_1, x_2) = (x, y)$

Ex $Y = (3,1)$, $A = (-1,2)$

$$(-1,2)\cdot\big((x_1,x_2) - (3,1)\big) = 0$$

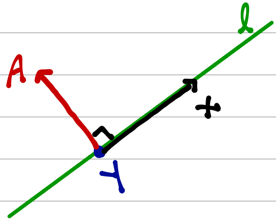
$$(-1,2)\cdot(x_1 - 3, x_2 - 1) = 0$$

$$-(x_1 - 3) + 2(x_2 - 1) = 0$$



$$(x_2 - 1) = \tfrac{1}{2}(x_1 - 3) \qquad y = \tfrac{1}{2}x - \tfrac{1}{2}$$

$\exists$ alternate version of normal form:

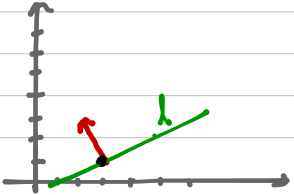


$A\cdot(X-Y)=0$

$A\cdot X - \underbrace{A\cdot Y}_{\text{constant } c} = 0$

$A\cdot X = c$

Ex $Y=(3,1)$, $A=(-1,2)$



$(-1,2)\cdot(X-(3,1))=0$

$(-1,2)\cdot(x,y)=(-1,2)\cdot(3,1)$

$-x+2y=-1$ etc.

$\left(y=\frac{1}{2}x-\frac{1}{2}\right)$

Warmup Questions

Prove: $\forall\, u,\ \|-u\| = \|u\|$

$\sqrt{(-u)\cdot(-u)} = \sqrt{u\cdot u}$

$\vdots$ (avoid)

$0 = 0$

Prove: $\|-u\|^2 = \|u\|$

Pf: $\|-u\|^2 = (-u)\cdot(-u)$

$= (-1)^2\, u\cdot u$

$= u\cdot u$

$= \|u\|^2$

Prove $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2x\cdot y$

"Alg. law of cosines"

$\|x-y\|^2 = (x-y)\cdot(x-y)$

$= x\cdot x - x\cdot y - y\cdot x + (-y)\cdot(-y)$

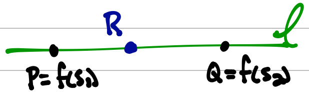
$= x\cdot x + y\cdot y - 2x\cdot y$

$= \|x\|^2 + \|y\|^2 - 2x\cdot y$

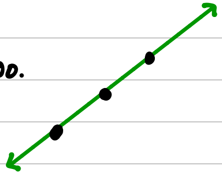
Betweeness

$/\!/\ B+sU$ $\quad$ $(s_1<s_2)$

Def/Prop Let $f(s)$ be eqn of line, $f(s_1)=P$, $f(s_2)=Q$. Then R is between P, Q if $\exists\, s_3$, $s_1<s_3<s_2$, s.t. $f(s_3)=R$.

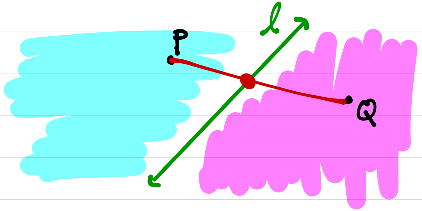


Corollary 1.22 Given 3 pts on a line, one must be b/w other two.

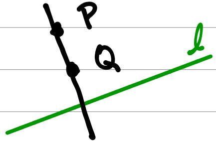


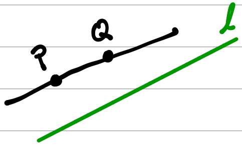
See book for further corollaries with 4+ points

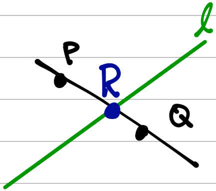
A line separates $\mathbb{R}^2$ into two "half planes."



<u>Clever Def</u> P, Q are on different sides of ℓ if $\exists\, R \in \ell$ between them.





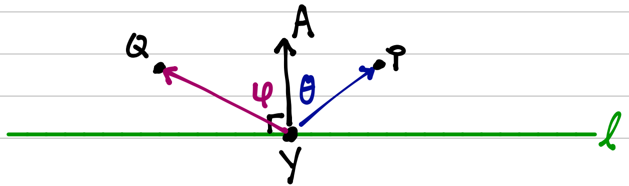


Prop 1.30 Let $\ell: A\cdot(X-Y)=0$ ($Y\in\ell$, $A\perp\ell$) and $P, Q\notin\ell$. Then P and Q are on same/opposite side of ℓ if $A\cdot(P-Y)$, $A\cdot(Q-Y)$ have same/opposite sign.

⚠ Book uses $A\cdot X=c$, compares $A\cdot P-c$, $A\cdot Q-c$.

⚠!! Would be "simple" (well, simpl<u>er</u>) if we had

$$\vec{a}\cdot\vec{b}=\|\vec{a}\|\cdot\|\vec{b}\|\cos\theta \quad \begin{array}{l} >0 \text{ for } \theta\in[0,\pi/2) \\ <0 \text{ for } \theta\in(\pi/2,\pi] \end{array}$$



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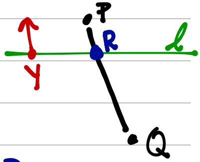
Pf Let $g(s)=A\cdot\big[\underbrace{(P+s(Q-P))}_{\text{line seg. }\overline{PQ}}-Y\big]$ for $\underline{0\le s\le 1}$



$g(s)=0$ for some s iff $\overline{PQ}$ intersects ℓ at pt R
($\Rightarrow$ P, Q opposite sides)

$$g(s)=\underset{\#}{A\cdot(P-Y)}+s\,\underset{\#}{A\cdot(Q-P)}=\cdots=b+sc \qquad (1+2s)$$

That's linear! min/max's at endpts, can be 0 if +/− (or −/+) at endpts

$$g(0)=A\cdot(P-Y)$$
$$g(1)=A\cdot(Q-Y)$$

day 2