Math 5335
Fall 2019
Exam 1
10/23/19
Time Limit: 75 Minutes

Name (Print): Key

This exam contains 7 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and **put your initials** on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you are applying a theorem, you should indicate this fact**, and explain why the theorem may be applied.
- **Do not trivialize a problem**. If you are asked to prove a theorem, you cannot just cite that theorem.
- **Organize your work** in a reasonable, tidy, and coherent way. Work that is disorganized and jumbled that lacks clear reasoning will receive little or no credit.
- **Unsupported answers will not receive full credit.** An answer must be supported by calculations, explanation, and/or algebraic work to receive full credit. Partial credit may be given to well-argued incorrect answers as well.
- If you need more space, use the back of the pages. **Clearly indicate when you have done this.**

Do not write in the table to the right.

Page	Points	Score
2	16	
3	13	
4	18	
5	21	
6	17	
7	15	
Total:	100	

You may use the following matrices and computations on the exam without defining or proving them.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad F_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}, \qquad R_\varphi R_\theta = R_{\varphi+\theta}, \qquad F_\varphi F_\theta = R_{2(\varphi-\theta)}$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Suppose ℓ forms an angle of θ with the horizontal. If $U \parallel \ell$, then $F_\theta U = U$. If $V \perp \ell$, then $F_\theta V = -V$.

1. (4 points) Let $U = (3,4)$ and $V = (5,-6)$. Compute the following quantities. (1 point each)

   (a) $U \cdot V$ $\quad (3,4)\cdot(5,-6) = 15 - 24 = -9$

   (b) $3U + 2V$ $\quad (9,12) + (10,-12) = (19,0)$

   (c) $U - V$ $\quad (-2,10)$

   (d) $\|U\|$ $\quad \sqrt{9+16} = 5$

2. Prove the following facts about vectors using methods from this class. Use our definition of the dot product, not the later characterization of $U \cdot V = \|U\|\,\|V\| \cos\theta$.

   (a) (6 points) Given two vectors U and V, prove $||U - V||^2 = ||U||^2 + ||V||^2 - 2\,U \cdot V$.

$$\|u+v\|^2 = (u+v)\cdot(u+v)$$
$$= u\cdot u + v\cdot v + 2u\cdot v$$
$$= \|u\|^2 + \|v\|^2 + 2u\cdot v$$

   (b) (6 points) Let U and V be unit vectors, and $t \in \mathbb{R}$. Prove $\|U + tV\| = \|V + tU\|$.

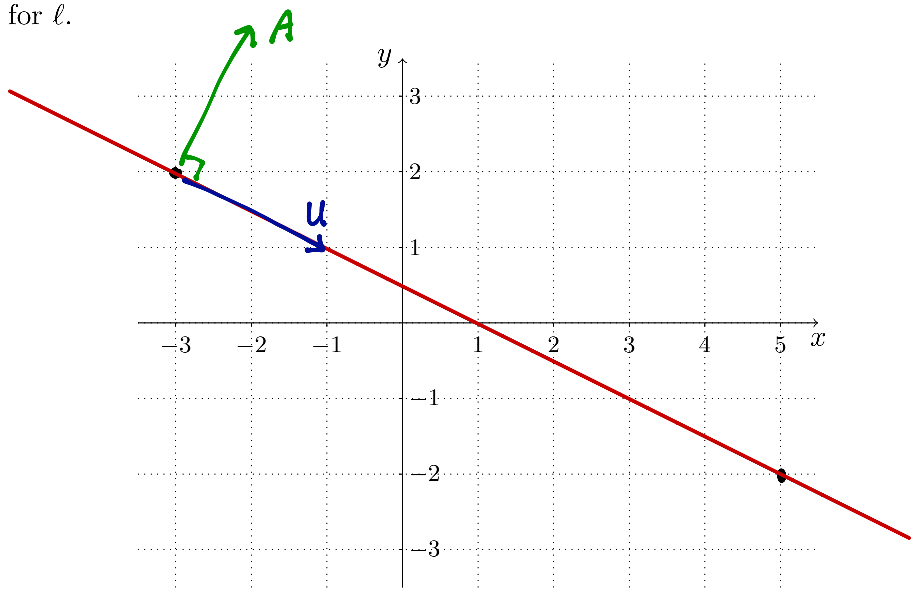
$$\|u+tv\|^2 = \|u\|^2 + \|tv\|^2 + 2t\,u\cdot v$$ (by above, or by doing similar work)
$$= 1 + t^2 + 2t\cdot u\cdot v$$

$$\|v+tu\| = \|v\|^2 + \|tu\|^2 + 2t\,u\cdot v$$
$$= 1 + t^2 + 2t\cdot u\cdot v$$

3. Consider the line $\ell = \{\, a(-3,2) + b(5,-2) \mid a + b = 1\}$

(a) (1 point) Sketch the line ℓ on the axes below.

(b) (4 points) Find a parametric equation for ℓ.



Many answers possible

$(-3,2) + s(2,-1),\ s \in \mathbb{R}$

$(5,-2) + t(-8,4),\ t \in \mathbb{R}$

(c) (4 points) Find a normal equation for ℓ.

$A \cdot (X - P) = 0$

$(1,2) \cdot (X - (-3,2)) = 0$

Or, multiplying out,

$(1,2) \cdot X - (1,2) \cdot (-3,2) = 0$

$(1,2) \cdot X = 1$

$(4,8) \cdot X = 4$

etc.

(d) (4 points) Find a parametric equation for the line k containing $(2,2)$ and perpendicular to ℓ.

$(2,2) + s(1,2),\ s \in \mathbb{R}$

4. Recall that $\arccos z = \displaystyle\int_z^1 \frac{1}{\sqrt{1-t^2}}dt$ and we define $\pi = \displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-t^2}}dt$.

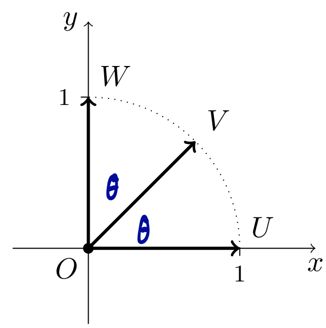
(a) (6 points) Use calculus to prove that $\arccos(0) = \pi/2$.

$$\arccos(0) = \int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\int_{-1}^1 \frac{1}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\pi$$

even integrand

(b) (6 points) In the picture below, $U = (1,0)$, $V = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $W = (0,1)$. Prove $|\angle UOV| = \pi/4$ using the methods of this course.

$U \cdot V = \frac{1}{\sqrt{2}} = V \cdot W$ so $|\angle UOV| = \displaystyle\int_{1/\sqrt{2}}^1 \frac{1}{\sqrt{1-t^2}}\,dt = |\angle VOW|$



$U \cdot W = 0 \Rightarrow |\angle UOW| = \displaystyle\int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt = \pi/2$ by above.

Thus $2\theta = \frac{\pi}{2} \Rightarrow \theta = |\angle UOV| = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

(c) (6 points) Let rays p and q emanate from a common vertex, with direction indicators $(4,-3)$ and $(1,2)$. Find $|\angle(p,q)|$. An answer with an integral is fine.

$$|\angle(p,q)| = \arccos\left(\frac{(4,-3)}{5}\cdot\frac{(1,2)}{\sqrt{5}}\right) = \arccos\left(\frac{4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}}\right) = \int_{\frac{-2}{5\sqrt{5}}}^1 \frac{1}{\sqrt{1-t^2}}\,dt$$

$U = \dfrac{(4,-3)}{\|(4,-3)\|}$

$V = \dfrac{(1,2)}{\|(1,2)\|}$

5. (a) (5 points) Let $\mathcal{U}$ and $\mathcal{V}$ be isometries of $\mathbb{R}^2$. Prove that the composition $\mathcal{U} \circ \mathcal{V}$ is an isometry.

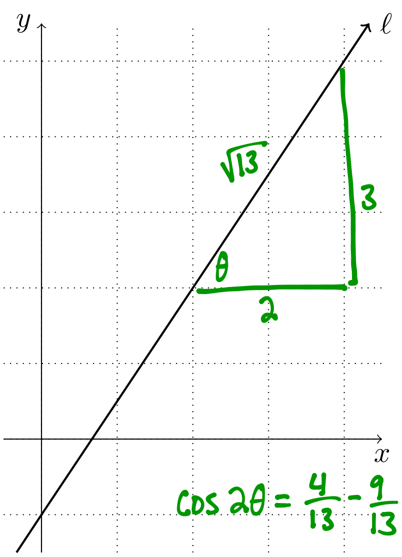
$$\forall\, P, Q, \quad \|\mathcal{U}(\mathcal{V}(P)) - \mathcal{U}(\mathcal{V}(Q))\| = \|\mathcal{V}(P) - \mathcal{V}(Q)\| = \|P - Q\|$$

(b) (8 points) Find the matrix formula $\mathcal{R}$ for the rotation of $\mathbb{R}^2$ by $3\pi/4$ about the point $(-1, 5)$. Your answer should include exact values in the matrix, not trig functions, but you do not need to multiply out your entire formula.

$$R_{\frac{3\pi}{4}} = \begin{bmatrix} c\,3\pi/4 & -s\,3\pi/4 \\ s\,3\pi/4 & c\,3\pi/4 \end{bmatrix}$$

$$\mathcal{R}(X) = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\left(X - \begin{bmatrix} -1 \\ 5 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

(c) (8 points) Let $\ell = \{(2, 2) + t(2, 3)\}$, where $t \in \mathbb{R}$. Find the matrix formula $\mathcal{M}_\ell(X)$ for the reflection across the line ℓ. Your answer should include exact values in the matrix, not trig functions, but you do not need to multiply out your entire formula.



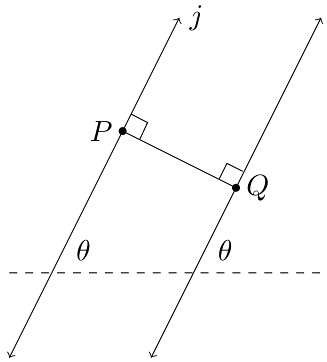
$$\mathcal{M}_\ell(X) = F_\theta(X - P) + P$$

$$= \begin{bmatrix} -5/13 & 12/13 \\ 12/13 & 5/13 \end{bmatrix}\left(X - \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\cos 2\theta = \frac{4}{13} - \frac{9}{13} = -\frac{5}{13}$$

$$\sin 2\theta = 2 \cdot \frac{2}{\sqrt{13}} \cdot \frac{3}{\sqrt{13}} = \frac{12}{13}$$

6. (8 points) In the diagram below, $j \parallel k$, and points $P \in j$ and $Q \in k$ are chosen so that the vector $Q - P$ is perpendicular to both j and l. The dotted line is horizontal, i.e. parallel to the x-axis. Use the methods of this course to prove $\mathcal{M}_k \circ \mathcal{M}_j$ is a translation by $V = 2(Q - P)$.



$$\mathcal{M}_k \circ \mathcal{M}_j(X) = F_\theta\left(F_\theta(X-P)+P-Q\right) + Q$$

$$= F_\theta F_\theta(X-P) + F_\theta(P-Q) + Q$$

$$= X - P + Q - P + Q$$

$$= X + 2Q - 2P$$

$F_\theta(P-Q) = -(P-Q)$
$= Q-P$

Since $P-Q \perp j,k$
("Useful Facts")

7. Let $\mathcal{C}_D(X)$ be rotation by π centered at a point D. (The book calls this a *central inversion*; hence the $\mathcal{C}$.)

(a) (3 points) Find a and b such that $\mathcal{C}_D(X) = aD + bX$.

$$R_\pi = \begin{bmatrix} c\pi & -s\pi \\ s\pi & c\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathcal{C}_D(X) = R_\pi(X-D)+D$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}(X-D)+D$$

$$= -X+D+D$$

$$= 2D - X$$

$a = 2$
$b = -1$

(b) (3 points) Prove that $\mathcal{C}_D$ is an involution, i.e. $\mathcal{C}_D$ is its own inverse.

$$\mathcal{C}_D(\mathcal{C}_D(X)) = \mathcal{C}_D(2D-X)$$

$$= 2D - (2D-X) = X$$

(c) (3 points) Suppose D and E are distinct points. Prove $\mathcal{C}_E \circ \mathcal{C}_D$ is a translation.

$$\mathcal{C}_E \circ \mathcal{C}_D(X) = \mathcal{C}_E(2D-X)$$

$$= 2E - (2D-X)$$

$$= X + 2(E-D)$$

8. (15 points) Indicate whether each statement is always **True** or could be **False** by circling the appropriate answer. Justify your answer with definitions, theorems and methods from this course. (If false, **be specific** with your explanation; give an example to show the statement is false.) (5 points each)

(a) If $\mathcal{U}$ is an isometry, then $\mathcal{U}(aP + bQ) = a\mathcal{U}(P) + b\mathcal{U}(Q)$ for all $a, b \in \mathbb{R}$ and points $P, Q \in \mathbb{R}^2$.

True **False**

HW - nonzero transl'ns

(And other possible answers)

(b) An isometry can never be a linear transformation.

True **False**

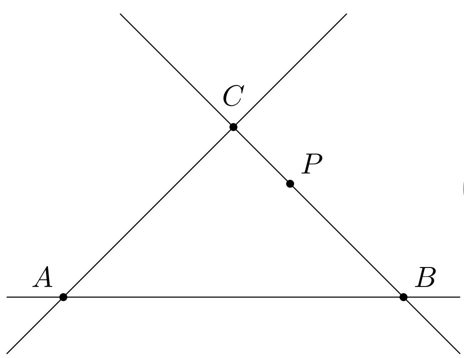
$\mathcal{U}(X) = MX + P = MX + \mathcal{U}(0)$

If $\mathcal{U}(0) = 0$, $\mathcal{U}$ a lin transl'n.

(e.g. rotation at origin. Or just $id(x)$.)

(c) If $P = (r, s, t)^{\triangle ABC}$ in the following picture, then $r = 0$.

True **False**



$(0,s,t)^{\triangle} = 0P + sB + tC \in \overleftrightarrow{BC}$ b/c $s+t=1$

(and $s,t>0$ b/c $P \in \overline{BC}$).