Geometry - Math 5335 Review of Lin Alg

To solve a system of eqn's such as

$$9s + 2t = 1 \qquad (1)$$
$$-5s - 5t = -5 \qquad (2)$$

you can use a few different methods:

(1) Substitution

In equation (1), you can solve for t:

$$2t = 1 - 9s$$
$$t = \tfrac{1}{2}(1 - 9s)$$

Now plug this in to eq'n (2):

$$-5s - 5\left[\tfrac{1}{2}(1 - 9s)\right] = -5$$

(algebra...) $\Rightarrow \frac{35}{2}s - \frac{5}{2} = -5$

(more algebra) $\boxed{s = -\frac{1}{7}}$

Now plug $s = -1/7$ into <u>both</u> eqn's to make sure each results in the same value of t:

$$9\left(-\tfrac{1}{7}\right) + 2t = 1 \Rightarrow \cdots \Rightarrow t = 8/7$$
$$-5\left(-\tfrac{1}{7}\right) - 5t = -5 \Rightarrow \cdots \Rightarrow t = 8/7$$

Thus there is a solution $(s, t) = (-1/7, 8/7)$

Substitution is hard for ≥ 2 eqns, 2 vars.....

(2) Elimination

Multiply eq'ns add add eqns together to eliminate variables.

$$\begin{aligned} 9s + 2t &= 1 \\ -5s - 5t &= -5 \end{aligned} \xrightarrow[\text{by } \frac{2}{5}]{\text{mult eqn 2}} \begin{aligned} 9s + 2t &= 1 \\ -2s - 2t &= -2 \end{aligned}$$

$$\xrightarrow[\text{w/ eqn1+eqn 2}]{\text{replace eqn 1}} \begin{aligned} &\boxed{7s + 0 = -1} \\ &-2s - 2t = -2 \end{aligned} \Longrightarrow s = -1/7.$$

Here I could plug $s = -1/7$ into eqn 2, or continue

$$\xrightarrow[\text{by } 1/7]{\text{mult eqn 1}} \begin{aligned} s + 0t &= -1/7 \\ -2s - 2t &= -2 \end{aligned} \xrightarrow[\substack{2\cdot(\text{eqn 1}) \\ + \text{eqn 2}}]{\substack{\text{replace} \\ \text{eqn 2 by}}} \begin{aligned} s + 0t &= -1/7 \\ 0s - 2t &= -16/7 \end{aligned}$$

$$\xrightarrow[\text{by } -2]{\text{divide eqn 2}} \boxed{\begin{aligned} s \qquad &= -1/7 \\ t &= 8/7 \end{aligned}}$$

(3) Matrix Methods

An $n \times m$ matrix is an array of numbers with <u>n rows</u>, <u>m columns</u>:

$$\underset{2\times2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \qquad \underset{2\times3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \qquad \underset{3\times1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

Two matrices can be multiplied only if the "inner dimensions" match when written side by side:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$2 \times 2 \quad 3 \times 2$ don't match; can't multiply!

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (1\cdot1 + 2\cdot3) & (1\cdot2+2\cdot4) \\ (3\cdot1+4\cdot3) & (3\cdot2+4\cdot4) \\ (5\cdot1+6\cdot3) & (5\cdot2+6\cdot4) \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \\ 23 & 34 \end{bmatrix}$$

$3 \times 2 \quad 2 \times 2$ ok! $\quad 3 \times 2$

If $A \cdot B = C$, the $(i,j)^{\text{th}}$ entry of C (i^{th} row, j^{th} column) is the dot product of the i^{th} row of A, j^{th} col of B.

⚠ In general, $AB \neq BA$!!

We use matrix mult. to "encode" systems of eqn's.

$$\underbrace{\begin{bmatrix} 9 & 2 \\ -5 & -5 \end{bmatrix}}_{2\times2} \underbrace{\begin{bmatrix} s \\ t \end{bmatrix}}_{2\times1} = \underbrace{\begin{bmatrix} 1 \\ -5 \end{bmatrix}}_{2\times1}$$ ← matrix form

$$\underbrace{\begin{bmatrix} (9s + 2t) \\ (-5s - 5t) \end{bmatrix}}_{2\times1} = \underbrace{\begin{bmatrix} 1 \\ -5 \end{bmatrix}}_{2\times1}$$ ← our original system

Recall the <u>inverse</u> of an $n \times n$ matrix A is another $n \times n$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} (= I_n$

(1's on diagonal, 0's elsewhere, the $n \times n$ <u>identity matrix</u>.)

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we know a formula: $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

⚠ This only exists if $ad-bc$ (also called the <u>determinant</u> of A, $|A|$, $\det A$) is not zero!!

In our system, let $A = \begin{bmatrix} 9 & 2 \\ -5 & -5 \end{bmatrix}$, so

$A^{-1} = \frac{1}{-35}\begin{bmatrix} -5 & -2 \\ 5 & 9 \end{bmatrix}$. We can solve the system

like this: $A\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

$$A^{-1}A\begin{bmatrix} s \\ t \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$I\begin{bmatrix} s \\ t \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \frac{1}{-35}\begin{bmatrix} -5 & -2 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{-35}\begin{bmatrix} 5 \\ -40 \end{bmatrix} = \begin{bmatrix} -5/35 \\ -40/-35 \end{bmatrix} = \begin{bmatrix} -1/7 \\ 8/7 \end{bmatrix}.$$

Sometimes we use an "augmented matrix" to do Elimination (Method ②) with less writing. Here's Method ② rewritten with matrices. (Can you follow each step?)

$$\left.\begin{aligned} 9s + 2t &= 1 \\ -5s - 5t &= -5 \end{aligned}\right\} \text{ becomes } \left[\begin{array}{cc|c} 9 & 2 & 1 \\ -5 & -5 & -5 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 9 & 2 & 1 \\ -2 & -2 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 7 & 0 & -1 \\ -2 & -2 & -2 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & -1/7 \\ -2 & -2 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & -1/7 \\ -1 & -1 & -1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & -1/7 \\ 0 & -1 & -8/7 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & -1/7 \\ 0 & 1 & 8/7 \end{array}\right]$$

$\Rightarrow$ $s = -1/7$, $t = 8/7$ is the solution.

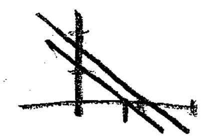
What can go wrong?

Ex 1

$$x + y = 1$$
$$2x + 2y = 3$$

(so $y = 1 - x$, subst. into eqn 2)

$$2x + 2(1-x) = 3 \Rightarrow 2x + 2 - 2x = 3$$
$$2 = 3 \quad ?!!?$$

Sometimes with substitution you get nonsense, or eqns which can't be satisfied. There are **NO** solutions. (geometrically, you have parallel lines w/ no intersection 

Ex 2

$$x + y = 1$$
$$2x + 2y = 2$$

By substitution,

$$2x + 2(1-x) = 2$$
$$2x + 2 - 2x = 2$$
$$2 = 2$$

Sometimes you get eqns which are true for any x and y! This is a case of infinitely many sol'ns. (geometrically, they're the same line)

Note in both cases, in matrix form, $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $\det(A) = 1 \cdot 2 - 1 \cdot 2 = 0$.