This is an open-book, -library, -internet take home exam. You are *not* allowed to collaborate; I am the only person you are allowed to consult. I'll hold my regular office hours, and in fact be available much of the rest of the time throughout the week. (I may not be in my office all the time, however, so it pays to email me and ask.) As always, you should explain your work, writing complete sentences with reasonably correct grammar.

Please note: it's not the intention that you copy any proof from some other textbook, but if you make heavy use of ideas from another source you should certainly cite it. (Having said that, I know what books and websites many people will go to and intentionally chose problems which aren't shown in those sources.)

Due Wednesday, 12/19 in my mailbox in Vincent 107 by 3:00pm.

FINAL EXAM

(1) (10 Points) In class we proved that $[-1, 1]$ is compact, and I pointed out that our work could be generalized to show that any closed and bounded set in $\mathbb{R}^n$ is compact. This is half of the celebrated Heine-Borel Theorem:

Theorem. *A subset C of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

   Prove the other half of the theorem: if a set is compact, it is closed and bounded. (Hint: prove the contrapositive.)

(2) (15 Points) Given a non-compact topological space X, the one-point compactification (or "Alexandroff" compactification) of X is the space $Y = X \cup \{\infty\}$, where ∞ is thought of as one extra "point" attached to the set. The topology on Y is the collection of all open sets in X together with all sets of the form $U \cup \{\infty\}$, where $U \subset X$ is open in X and $X \setminus U$ is compact.
   (a) Let $X = \mathbb{R}$ and Y denote the one-point compactification of X. Describe all of the open sets in Y. (Because you know specifically what X is, your description should contain more detail than simply repeating the above definition.)
   (b) Prove that Y is homeomorphic to S^1. You proof should not be based on a picture or handwaving, but be a rigorous argument using definitions of homeomorphism, continuous, etc.

(3) (15 Points) Identify the surface obtained when the pairs of edges of a triangular annulus are glues together as shown in Figure 3.44 (on page 118 of Topology Now).

(4) (15 Points)
   (a) Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup A_n$ is connected.

(b) Recall the finite complement topology, in which a set is closed if and only if it is finite. (Equivalently, a set is open if and only if its complement is finite.) Show that if X is an infinite set, it is connected in the finite complement topology.

(5) (15 Points) Let $X = [0, 1]$, and define $Y \subset \mathbb{R}^2$ to be the union of the following sets:

$$\{(x, 0) \,|\, 0 \le x \le 1\}$$
$$\{(1/n, y) \,|\, 0 \le y \le 1, \, n \in \mathbb{N}\}$$

(Y is a variant of a set called the "Topologist's Comb." To the extent possible, try to draw a picture of Y.)

(a) Construct a bijection $f : X \to Y$.

(b) Prove X and Y are not homeomorphic.

(6) (15 Points)

(a) Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

(b) Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$, $\overline{A \times B} = \overline{A} \times \overline{B}$.

(7) (15 Points) Recall that a pseudometric on a set X is a function $d : X \times X \to \mathbb{R}$ which satisfies all the conditions of a metric, except that it is possible to have $d(x, y) = 0$ for $x \ne y$. Choosing $X = \mathbb{R}^2$, let

$$d\left((x_1, y_1), \, (x_2, y_2) \right) = |x_1 - x_2|$$

(a) Prove that d is a pseudometric but not a metric.

(b) As with a metric, let $B_r(x) = \{y \in X \,|\, d(x, y) < r\}$. Prove that the collection of sets $\mathcal{B} = \{B_r(x) \,|\, x \in X, \, r > 0\}$ forms a basis for a topology on $\mathbb{R}^2$. How does this topology compare with the standard topology on the Euclidean plane?

(8) (5 Points) As you all recall, the department's curriculum committee asked me to run something of an experiment this year using the Topology Now! textbook and modifying the list of topics covered. For the record, after talking to the other topologists in the department, I'm planning on recommending that we keep the classification of surfaces in the class, but increase the attention given to certain classical and important ideas like connectedness and compactness. Because those each receive very little coverage in Topology Now! that might mean recommending a different textbook. Having said that, for 5 points I'd appreciate any comments you have on the textbook.

Additionally, it would be useful for you to describe (a) your current year/major, (b) your future plans [specifically, grad school or not?] and (c) if you have any comments about what topics you most enjoyed or missed. But you aren't required to answer these more "personal" questions to get the five points.