Your first exam is scheduled for Friday, October 19th. Officially it covers Chapter 1 and Section 7.1 of your textbook, although you should recall that there are certain things (like the characterizations of continuity) which I added to your textbook's coverage. You should know definitions of basic terms—everything from injective to metric and open set—as well as be able to use them correctly in different settings. The problems are similar in nature to homework problems, examples in class, and examples in your book. You won't be asked to prove anything which hasn't been done in class, perhaps with minor changes in intervals, flipping unions and intersections, etc.

Don't trivialize a problem. For example, if I ask you to prove that a function from $\mathbb{R}$ to $\mathbb{R}$ is continuous, you shouldn't say "I can draw the graph without lifting my pencil." In the context of this class you need to do a more careful proof of continuity, whether with ϵ and δ, sequences, or inverse images of open sets. This holds true for other problems; if I proved something carefully in class using a particular theorem, you can't simply say that it is "clearly true" on an exam and expect to get full credit.

I've included some sample problems below. The problems here and the topics they cover are not in bijection with the problems on the test; rather, they cover topics which I think are important and/or I know have caused some people difficulty.

1. *This problem is a bit abstract and full of tedious details, but is good review if you're shaky with set theory and function definitions.* Let $f : X \to Y$. Which of the following are always true, and which are false in general? For those that are false, are there any conditions you could add (f injective? surjective?) to make them true? Here A is a subset of X, and A_α represents some family (finite, countable, or uncountable) of subsets of X. (Same for the B's and Y.)
   - $f(\cup_\alpha A_\alpha) = \bigcup_\alpha f(A_\alpha)$
   - $f(\cap_\alpha A_\alpha) = \bigcap_\alpha f(A_\alpha)$
   - $f(X - A) = B - f(A)$
   - $f^{-1}(\cup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha)$
   - $f^{-1}(\cap_\alpha B_\alpha) = \bigcap_\alpha f^{-1}(B_\alpha)$
   - $f(Y - B) = X - f(B)$

2. Which of the following are equivalence relations on $\mathbb{R}$? For each that is, describe the equivalence class C_x of $x \in \mathbb{R}$.
   - $a \sim b$ iff $a - b \in \mathbb{Q}$
   - $a \sim b$ iff $a - b \notin \mathbb{Q}$
   - $a \sim b$ iff $a - b \in \mathbb{Z}$
   - $a \sim b$ iff $|a - b| \leq 1$

3. Prove that $\mathbb{R}$ is homeomorphic to the infinite spiral which is the graph of $(\cos t, \sin t, t)$, where $t \in \mathbb{R}$.

4. Find a bijection from $X = \{x^2 + y^2 < 1\} \subset \mathbb{R}^2$ to $Y = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$. Prove that X and Y are not homeomorphic.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x + y$. Prove f is continuous, where $\mathbb{R}$ has the standard metric and $\mathbb{R}^2$ has either the standard metric or the taxi cab metric.

6. Let (X, d) be a metric space and k a positive real number. Define $d_k(x, y) = k \cdot d(x, y)$. Prove that (X, d_k) is a metric space.

7. Describe an ambient isotopy which takes the region $X \subset \mathbb{R}^2$ described in polar coordinates by $0 \leq r < 1$, $0 \leq \theta \leq 3\pi/2$ to the region $Y = [-1, 0) \times (0, 1) \subset \mathbb{R}^2$.

8. Let $\mathbb{Z}$ be the set of integers, and let p be a fixed positive prime number. Given distinct integers $m, n \in \mathbb{Z}$ it can be shown there is a unique integer $t = t(m, n)$ such that $m - n = p^t k$ where k is an integer not divisible by p. Define a function $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by $d(m, m) = 0$ and

$$d(m, n) = \frac{1}{p^t}$$

for $m \neq n$. (So m and n are "close" if their difference is divisible by a large power of p.) Prove that $(\mathbb{Z}, d)$ is a metric space.

For those of you who are interested, this leads to the *p-adic integers* and, with a bit more work, the *p-adic numbers*.