Comments on the solutions to the study guide problems; let me know if you have any questions or spot any typos.

1. *Show that the connected sum $T^2 \# T^2$ of two tori can be represented by a octagonal disk with sides identified according to the word $aba^{-1}b^{-1}cdc^{-1}d^{-1}$.*

   This was an example done in class on a pink worksheet. You can also find it online, for example at at Google Books. Here's another such diagram, although the picture here organizes the edges in a different fashion.

2. (a) *Show that the connected sum $P^2 \# P^2$ of two projective planes can be represented as a square disk with sides identified according to the word aabb.*

   (b) *Prove that the $P^2 \# P^2$ is homeomorphic to a Klein Bottle.*

   This is essentially the same as Example 3.15 and Problem 7 in section 3.2, for which I handed out a solution in class on Wednesday.

3. *Identify the resulting surfaces (up to homeomorphism) if the edges of a square are identified according to the following words.*

   (a) $aa^{-1}bb^{-1}$

   (b) $abab$

   (c) $a^{-1}bab$

   In order: the sphere S^2, the projective plane P^2, and the Klein Bottle K^2.

4. *Let X be a topological space, and $A \subset X$ have the relative (or "subspace") topology. Prove the following theorems:*

   (This is similar to problem K1 on Homework 4; solutions are posted online. It can be very helpful to draw an example in $\mathbb{R}$ or $\mathbb{R}^2$.)

   Theorem A set $C \subset A$ is closed in A if and only if it is the intersection of A with a closed set of X.

   Proof: Let $C = A \cap D$ for some set D which is closed in X; that means $D = X - U$ for an open set U in X. (Note that, by definition, $A \cap U$ is open in A.) We wish to show that $A - C$ is open in A, so that C is closed in A. But $A - C$ (the points in A which are not in C) is precisely equal to the intersection of A and $X - D$, so:

   $$A - C = A \cap (X - D) = A \cap U$$

   which is open in A.

   In the other direction, suppose C is closed in A, so $A - C$ is open in A. Hence $A - C = A \cap U$ for an open set U in X. That means $X - U$ is closed in X, and

   $$C = A \cap (X - U)$$

   is therefore the intersection of A with a closed set of X.

   Theorem If $B \subset A$ and $\overline{B}$ denotes the closure of B in X, then the closure of B in A equals $\overline{B} \cap A$.

   Proof: To keep things straight, write $\mathrm{cl}_A(B)$ and $\mathrm{cl}_X(B)$ for the closure of B in A and X respectively, so I want to prove that $\mathrm{cl}_A(B) = \mathrm{cl}_X(B) \cap A$. Recall that, in either space, the closure of B

is the smallest closed set (in that space) which contains B. (So if $B \subset C$ for a closed set C, the closure of B is contained in C.)

In one direction, $\mathrm{cl}_X(B)$ is closed in X so, by the previous theorem, $\mathrm{cl}_X(B) \cap A$ is closed in A. It's also true that $B \subset \mathrm{cl}_X(B)$ and $B \subset \mathrm{cl}_X(B) \cap A$. (The latter follows from the former because B is wholly contained in A.) By the remarks at the end of the previous paragraph, $\mathrm{cl}_A(B) \subset \mathrm{cl}_X(B) \cap A$.

In the other direction, $\mathrm{cl}_A(B)$ is closed in A, so (by the previous theorem again) it equals $C \cap A$ for some closed set C in X. Since C is closed in X and must contain B, $\mathrm{cl}_X(B) \subset C$. Hence

$$\mathrm{cl}_X(B) \cap A \subset C \cap A \subset \mathrm{cl}_A(B)$$

Since $\mathrm{cl}_A(B) \subset \mathrm{cl}_X(B) \cap A$ and $\mathrm{cl}_X(B) \cap A \subset \mathrm{cl}_A(B)$, we must have $\mathrm{cl}_A(B) = \mathrm{cl}_X(B) \cap A$.

5. *Consider $\mathbb{Q}$ as a subset of $\mathbb{R}$, where the real numbers are given the standard topology. Prove that the closure of $\mathbb{Q}$ equals all of $\mathbb{R}$.*

   It's tricky to determine the smallest closed set in $\mathbb{R}$ containing $\mathbb{Q}$, so we'll use one of the other characterizations: $\overline{\mathbb{Q}} = \mathbb{Q}^i \cup \mathbb{Q}^b$ or $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}'$, where $\mathbb{Q}^b$ is the set of $\mathbb{Q}$'s boundary points and $\mathbb{Q}'$ is the set of $\mathbb{Q}$'s limit (or "accumulation") points. Here are the two approaches:

   $\mathbb{Q}^i \cup \mathbb{Q}^b$: Every real number x is a boundary point of $\mathbb{Q}$, because any open set in $\mathbb{R}$ containing x contains *both* rational and irrational numbers[1]—i.e. every open set containing x has non-empty intersection with both $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$. Hence we don't have to figure out what $\mathbb{Q}^i$ is; $\mathbb{Q}^i \cup \mathbb{Q}^b$ is automatically $\mathbb{R}$.

   $\mathbb{Q} \cup \mathbb{Q}'$: Every real number x is also an accumulation point of $\mathbb{Q}$, which makes $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$. The reason every number is an accumulation point is quite similar to the previous paragraph. Let $x \in \mathbb{R}$. Then every open set containing x also contains rational numbers which are not equal to x.

6. *Let $X = \mathbb{N}$. Let $\mathcal{S} = \{X_i \mid i = 1, 2, 3, \ldots\}$ denote the collection of subsets X_i, where $X_i = \mathbb{N} - \{i\}$. Prove that $\mathcal{S}$ is a subbasis on X. Describe the resulting basis and topology on X.*

   The only requirement for $\mathcal{S}$ to be a subbasis is that for any $x \in X = \mathbb{N}$, there is a set $S \in \mathcal{S}$ such that $x \in S$. This is definitely true. 1, for example is in every set but X_1:

$$\begin{aligned} X_2 &= \{2, 3, 4, 5, 6, \ldots\} \\ X_3 &= \{1, 2, 4, 5, 6, \ldots\} \\ &\text{etc.} \end{aligned}$$

   (In fact, $n \in X_i$ for $i \neq n$.) To make a basis from a subbasis, you do finite intersections of the sets in the subbasis. In this case,

$$X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_k} = \mathbb{N} - \{i_1, i_2, \ldots, i_k\}$$

   In other words, finite intersections result in basis sets which have all but finitely many of the natural numbers. The open sets in the topology are unions of these sets, and it's the "finite complement" topology we discussed in class: a set $U \subset \mathbb{N}$ is open if and only if $U^c = \mathbb{N} - U$ is finite.

[1] This follows from the fact that every open interval contains both rational and irrational numbers, which can be proven rigorously. In this class you can use it without proof.

7. *Let $\mathcal{B} = \{[a, b) \mid a < b,\ a, b \in \mathbb{R}\}$, where $[a, b) = \{x \mid a \leq x < b\}$ as usual. Prove that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. Is this topology finer or coarser than the standard topology on $\mathbb{R}$? (In other words, does it have more or fewer open sets than the standard topology?)*

   The proof that this is a basis was Problem A on Homework 4, and a solution is online. The topology (known as the Sorgenfrey topology on the real line) is much finer than the standard topology. This is because (a) everything that's open in the standard topology is open in the Sorgenfrey topology, and (b) there are sets which are in the Sorgenfrey topology which are *not* open in the standard topology.

   To prove (b), note that $[0, 1)$ is open in the Sorgenfrey topology, but not in the standard topology.

   Top prove (a), we recall that open sets in the standard topology are unions of intervals (a, b), so if we can show (a, b) is open in the Sorgenfrey topology, we're done. But

$$\bigcup_{n=1}^{\infty} [a + 1/n, b) = (a, b)$$

   is a union of open sets, hence open.

8. *Prove the statement, or find a counter-example to show it is false.*

   (a) $\overline{A} \cap \overline{B} = \overline{A \cap B}$

   This is false; let $A = (0, 1)$ and $B = (1, 2)$. Then $\overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\}$, but $\overline{A \cap B} = \overline{\emptyset} = \emptyset$.

   (b) $\overline{A} \cup \overline{B} = \overline{A \cup B}$

   This is true, and is part 4 of Proposition 3.2 in Kahn's book, which we proved in class. In this proof it's helpful to recall that $\overline{A \cup B}$ is the smallest closed set which containts $A \cup B$. Put differently, if C is a closed set which contains $A \cup B$ (that is, $A \cup B \subset C$), then $\overline{A \cup B} \subset C$.

   First note that $A \subset \overline{A}$ and $B \subset \overline{B}$ immediately implies that $A \cup B \subset \overline{A} \cup \overline{B}$. Since $\overline{A} \cup \overline{B}$ is a closed set, it must be true that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ by the reasons in the previous paragraph.

   In the other direction, $A \subset A \cup B$, so $\overline{A} \subset \overline{A \cup B}$. Similarly, $\overline{B} \subset \overline{A \cup B}$. Together those facts mean $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

   Since $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ and $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$, we have shown that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.