These solutions aren't intended to be completely comprehensive, but should at least give you an idea of how to approach each problem. Some of the problems are best explained with a blackboard or a piece of paper, so if you have any trouble deciphering the solutions, stop by my office and ask. Also, let me know if you spot any typos and I'll update them as soon as I can.

Section 1.3 #11: **(a):** With a discrete space X as the domain, you can always use $\delta = 1/2$ (or any other number between 0 and 1). Then, for a given $a \in X$ and $\epsilon > 0$, $d(x, a) < \delta$ forces $x = a$, in which case $|f(x) - f(a)| = 0 < \epsilon$.

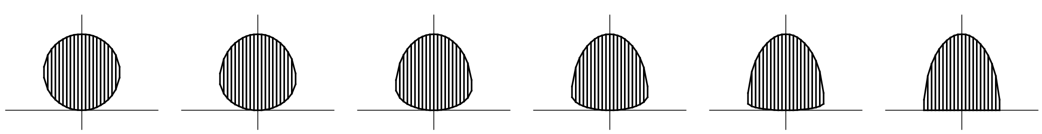
(b): As discussed extensively in class, this direction is a bit trickier. Let $f : \mathbb{R} \to Y$ be continuous at every $a \in \mathbb{R}$, where Y is a discrete space. Choose any $a \in \mathbb{R}$ and specify $\epsilon = 1/2$. Then there exists a δ so that $|x - a| < \delta$ forces $d(f(x), f(a)) < \epsilon = 1/2$, which in fact means $f(x) = f(a)$ for all $x \in B_\delta(a)$. Because this is true for any a, we can say that f is "locally constant." We still need to show it's globally constant; near a we have $f(x) = f(a)$, and near b we have $f(x) = f(b)$, and so on, but what if $f(a) \neq f(b)$?

So assume that f is not constant. Then there are numbers $a < b$ with $f(a) \neq f(b)$. Let

$$c = \sup\{x \in [a, b] | f(x) = f(a)\}$$

By the work above, there exists a δ_c such that $f(x) = f(c)$ for all $x \in (c - \delta_c, c + \delta_c)$. This quickly leads to a contradiction, however, because by the definition of c, $f(x) = f(a)$ for all $x \in [a, c)$, which includes the points in $(c - \delta_c, c)$. But to the right of c on the number line, the values of f must change (by the definition of c, which is impossible since f must be constant throughout $(c - \delta_c, c + \delta_c)$. Hence our assumption that f is not constant is false, and we're done.

Section 1.4 #9: Here's a graphical representation of this sliding process. Remember that the function will go straight from the first picture to the last picture; the intermediate stages aren't necessary. (But they're helpful when trying to understand what the function is doing.)



(a): One way to show this is to look at the height of each vertical chord. In the circle this height can be described simply (if not elegantly) as the height of the top half minus the height of the bottom half:

$$h_c(x) = 1 + \sqrt{1 - x^2} - (1 - \sqrt{1 - x^2}) = 2\sqrt{1 - x^2}$$

The height of a vertical chord in the ellipse for a given value of x is given by solving the equation for y:

$$h_e(x) = \sqrt{4(1 - x^2)}$$

These values agree for each $x \in [-1, 1]$.

(b): Given a point (x, y), the amount of vertical sliding is dependent only on x. We slide a point down equal to the distance from the x-axis to the bottom of the circle. This height is given by $1 - \sqrt{1 - x^2}$.

(c): Using the information from (b), we have

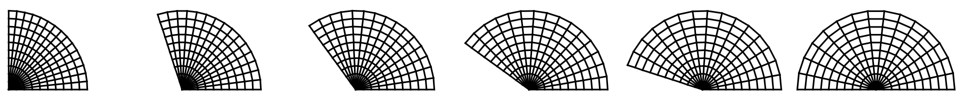
$$s(x, y) = (x, y - (1 - \sqrt{1 - x^2}))$$

(d): The inverse function of s is given by

$$t(x, y) = (x, y + (1 + \sqrt{1 - x^2})$$

I will leave it to you (using arguments from a Calculus book, for example) to show that s and t are both continuous bijections, and therefore homeomorphisms.

Section 1.4 #14: In part (a), "unfold" the quarter plane like a fan:



If you're having trouble understanding this, it's most easily described with polar coordinates:

$$h : \mathbb{R}^2 \to \mathbb{R}^2, \qquad h(r, \theta) = (r, 2\theta)$$

You may not map a point (x, y) to *both* (x, y) and $(-x, y)$. That's not a function.

Part (b) is very similar. The quarter-space could be described in spherical coordinates as $\rho \geq 0$, $\theta \in [0, \pi]$, $\phi \in [0, \pi/2]$. The "unfolding" function is then:

$$h : \mathbb{R}^3 \to \mathbb{R}^3, \qquad h(\rho, \theta, \phi) = (\rho, \theta, 2\phi)$$

Section 1.5 #3: For (a) and (b) I'll indicate a possible homeomorphism you could use; I'll leave it to you to show that this is a continuous bijection with continuous inverse. (Feel free to ask questions.)

(a): $f : (a, b) \to (0, 1)$, $f(x) = (x - a)/(b - a)$

(b): $f : (0, 1) \to (a, \infty)$, $f(x) = \tan\left(\frac{\pi}{2}x\right) + a$ (other half similar)

(c): Shown in class using the function (and inverse) from Example 1.6 on page 7.

(d): Any two open intervals are homeomorphic to $\mathbb{R}$ and hence homeomorphic to each other, since "homeomorphic to" is an equivalence relation.

Section 1.5 #8: In (a) and (b), pictures can be very instructive but don't constitute a proof by themselves. You needed to write enough words to convince the grader that the spaces were [or were not] path connected.

(a): You can modify the argument in Example 1.45 to show that a path $\alpha : [0, 1] \to (-\infty, a) \cup (a, \infty)$ cannot "jump over the gap" caused by removing the point $a \in \mathbb{R}$.

(b): Consider the Euclidean plane with a point P removed, $X = \mathbb{R}^2 - P$. Any two points in X can almost certainly be joined by a straight line segment; this will only fail if P happens to be on that segment. In this case another path—such as a circular arc—can still be constructed between the two points, so X is path connected.

(c): If h were a homeomorphism from $\mathbb{R}$ to $\mathbb{R}^2$, then the restriction of h to $\mathbb{R}$ minus a point would be a homeomorphism onto its image, which would necessarily be $\mathbb{R}^2$ minus a point. But the domain of this restricted homeomorphism would have two path components, whereas the image would have one. This is impossible, since the number of path components is a topological invariant.

Section 1.5 #14: Let $c = \sup A$. The crucial observation here is that every intersection $A \cap (c - 1/n, c) \neq \emptyset$; this is a fancy way of saying there are points in A which are arbitrarilty close to c. Why is this fact true? Suppose not; then there would be an n such that $(c - 1/n, c)$ contains no points in A. But this would imply that $c - 1/n$ is an upper bound for A – it's greater than or equal to all the numbers in A. This would contradict the fact that c is the *least* upper bound for A.

Thus we can construct a sequence by choosing a_n to be any number in the non-empty intersection $A \cap (c - 1/n, c)$. This sequence certainly converges to c, because a_n is within $1/n$ of c.

A: In order:

(1) $X \cup Y$ is not necessarily path connected, because they may be disjoint. (Example: $[0, 1]$ and $[2, 3]$.)

(2) $X \cap Y$ is not necessarily path connected, because you could have two sets which intersect in two disjoint components. (Example: take the parts of the unit circle for which $y \geq 0$ and $y \leq 0$.)

(3) $X \times Y$ is path connected. To construct a path from (x_1, y_1) to (x_2, y_2), start with any path α from x_1 to x_2 in X, and any path β from y_1 to y_2 in Y. Then the function from $[0, 1] \rightarrow X \times Y$ which maps t to $(\alpha(t), \beta(t)$ is a path.

(Note that we didn't really prove the continuity of this function until class on 10/8, but this point could be glossed over in the homework.)

There are other possible answers; for example, during the first half of the interval $[0, 1]$ you could use α to move from (x_1, y_1) to (x_2, y_1), and then use β to move from (x_2, y_1) to $(x_2, y_2$ during the second half.

B: This probem wasn't officially graded, which is probably good. Looking through I saw a few arguments which had the right idea but were incorrect in their logic. Some went like this:

"Consider the curve A on the torus T and the curve B on the two-holed torus S. Since A and B are homeomorphic, we must have $T - A$ and $S - B$ homeomorphic as well."

The above statement is *not* true, unless you've shown that a particular homeomorphism from T to S maps A onto B, in which case the restriction of that homeomorhpism will map $T - A$ to $S - B$. If you don't understand what I mean, ask me for a counterexample.

A much better approach would be: "Consider the curve B on the two-holed torus. Removing B leaves two path components. [You can figure out what B might be!] Any homeomorphism from S to T would have to match B with a curve on the torus with a similar property. Since there is no curve on the torus whose removal results in two path components, S and T cannot be homeomorphic."

(Incidentally, is the fact that there's no such curve on the torus true? Easy to prove?)