These solutions aren't comprehensive, so please ask if you have questions or spot a typo.

7.2 #4 (a) Take any two points x, y in a metric space X. Let $r = d(x, y)/2$. Then $U = B_r(x)$ and $V = B_r(y)$ are disjoint open sets satisfying the Hausdorff Axiom.

(b) (Following the hint given in class and via email.) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite topological space satisfying the Hausdorff Axiom. I claim first that each element x_i is a closed set by itself. I'll prove this for x_1.

For each $i = 2, \ldots n$ apply the Hausdorff Axiom to get a set V_i which is open, contains x_i and does **not** contain x_1. Then $V = \bigcup_{i=2}^{n} V_i$ is an open set whose complement is $\{x_1\}$. Hence, by definition, x_1 is closed.

A similar argument shows that each $\{x_i\}$ is a closed set. By the proposition I proved in class, if each single-element subset of a ***finite*** topological space is closed, then the space must have the discrete topology.

7.2 #10 $\overline{A}$ is the closed unit disk; A° is the open unit disk; the boundary of A is the unit circle.

7.2 #11 (a) A topology must contain finite intersections of any sets in the topology, so we have to add the following sets:

$$\{a, b, c\} \cap \{c, d\} = \{c\}$$
$$\{a, b, c\} \cap \{a, c, d\} = \{a, c\}$$

Thus the list of open sets is :

$$\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, c, d\}, \{c, d\}, \{c\}, \{a, c\}$$

(b) Taking complements, we see that the list of *closed* sets is:

$$X, \emptyset, \{b, c, d, e\}, \{d, e\}, \{e\}, \{b, e\}, \{a, b, e\}, \{a, b, d, e\}, \{b, d, e\}$$

The closure of $A = \{a, d, e\}$ is the intersection of all closed sets containing A. In this case only two closed sets contain A, so

$$\overline{A} = X \cap .\{a, b, d, e\} = \{a, b, c, d, e\} \cap \{a, b, d, e\} = \{a, b, d, e\}$$

The interior of A is the union of all open sets contained inside A. In this case there's only one open set contained in A, so $A^\circ = \{a\}$.

We also have $\text{bd}(A) = \overline{A} - A^\circ = \{a, b, d, e\} - \{a\} = \{b, d, e\}$

7.2 #12 I won't write out a full solution to this problem, because most of the problems occured with the first or second operation. Note that

$$\overline{A} = [0, 3] \cup \{4\}$$
$$A^\circ = (0, 1) \cup (1, 2)$$

In particular, the set $(2,3) \cap \mathbb{Q}$ is not closed in $\mathbb{R}$, and its closure is the entire closed interval $[2,3]$, not just the rationals in that interval (i.e. $[2,3] \cap \mathbb{Q}$). This is because every irrational number in $(2,3)$ is a boundary point (and also a limit point) of $(2,3) \cap \mathbb{Q}$ and therefore must be contained in its closure.

Why is this true? Let $A = (2,3) \cap \mathbb{Q}$ and x an irrational number between 2 and 3. Then every open set containing x will contain both rationals and irrationals. If you look at the definition of boundary point or limit point, you'll see that x satisfies both.

7.5 #2 As with the previous problem, I won't write out as many details as I possibly could here, but enough for you to understand the basic idea.

(a) Each interior point of the disc makes up its own equivalence class. There is one more equivalence class, which consists of all the points on the unit circle.

(b) This part is easier to explain in person and won't be necessary for the exam, so ask me in person.

(c) $X/\sim$ is homeomorphic to the sphere S^2.

A We need to verify two conditions. First, it's clear that every $x \in \mathbb{R}$ is contained in at least one set in $\mathcal{B}$. (Every point is contained in infinitely many of these "half open" intervals, in fact.) Second, suppose $B_1 = [a_1, b_1), B_2 = [a_2, b_2) \in \mathcal{B}$ such that

$$x \in B_1 \cap B_2 = [a_1, b_1) \cap [a_2, b_2)$$

Among other things, this means x is larger than (or equal to) both a_1 and a_2, and smaller than both b_1 and b_2. Let $B_3 = [\max\{a_1, a_2\}, \min\{b_1, b_2\})$. Then

$$x \in B_3 \subset B_1 \cap B_2$$

as required. Hence $\mathcal{B}$ is a basis.

K1 The proof isn't long, but requires keeping a number of sets straight. Ask me for help if you have trouble following this; often it helps to draw a few examples using the real number line, although that won't constitute a general proof.

First assume that B is closed in X. That means $U = B^C = X - B$ is an open set in X. By definition of the relative topology, $A \cap U$ is an open set in A. But note that B is closed in A if $A - B = A \cap U$ is open in A. Hence B is closed in A.

In the other direction, assume B is closed in A, so that $A - B$ is open in A. By definition of the relative topology, $A - B = A \cap U$ for some open set U in X. The set $X - U$ is closed in X, but $X - U = B$, so B is closed in X.

K2 Consider the open ball $B = B_1(0,-1)$. Because the point $(0,-1)$ is "isolated" from the rest of the set, $B_1(0,-1) = \{(x,y) \,|\, \sqrt{x^2 + (y+1)^2} < 1\}$ only contains the point $(0,-1)$ itself! But this is a closed set itself, so $\overline{B} = B$! On the other hand, the "closed disk" $B_1(0,-1) = \{(x,y) \,|\, \sqrt{x^2 + (y+1)^2} \leq 1\}$ would also include the point $(0,0)$.