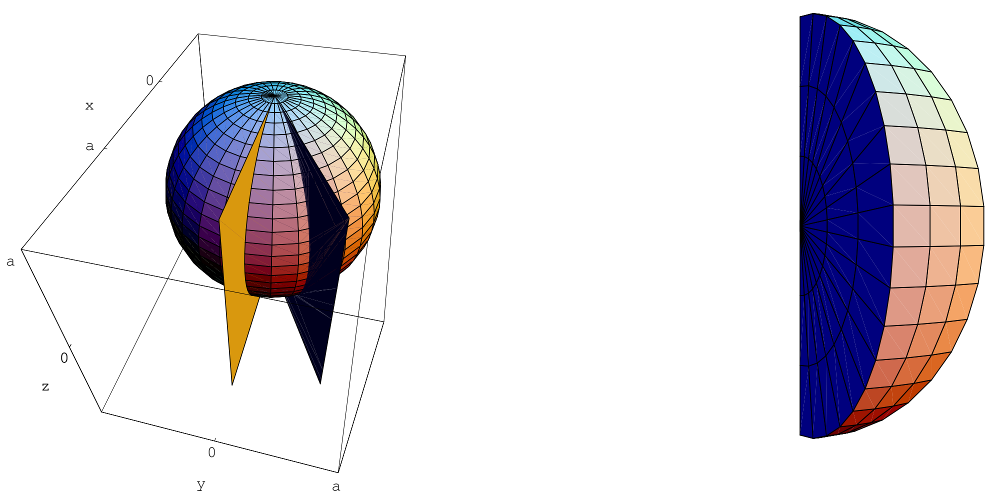
This writeup should give you an idea of what I would look for when grading this problem. It is is *not* intended to be a perfect solution. In particular, I would expect a better picture, with the planes $\theta = 0$ and $\theta = \pi/6$ labeled. While I could draw better pictures in two minutes by hand, it is quite time-consuming with the computer. I've chosen to leave it as is. If you are in doubt about what you should include in a good picture, talk to your workshop instructor.

#35. Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.

This problem is considerably easier if we put some care into our choices of spheres and planes. First, it's generally simplest to center a sphere at the origin. Next, we should try to choose the easiest possible equations for the two planes.

Because we are working with a piece of a sphere, it seems likely that we should work with spherical coordinates; with that in mind, the simplest choices for the planes are probably those described by $\theta = 0$ and $\theta = \pi/6$. Figure 1(a) shows a sphere of radius a together with these planes; we're interested in the apple-slice-shaped piece inbetween the planes. We'll call this solid region W. A closup view is shown in Figure 1(b).



(a) (b)

FIGURE 1. (a) The intersection of a sphere and two planes (b) The wedge W cut out of the sphere

By examining the pictures, we can see that the wedge W is described by the following inequalities in spherical coordinates:

$$\begin{aligned} 0 \leq &\rho \leq a \\ 0 \leq &\theta \leq \pi/6 \\ 0 \leq &\phi \leq \pi \end{aligned}$$

Hence the volume is given by the integral

$$\begin{aligned} \int\int\int_W dV &= \int_0^{\pi/6} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \quad d\rho d\phi d\theta \\ &= \int_0^{\pi/6} \int_0^{\pi} \frac{a^3}{3} \sin\phi \quad d\phi d\theta \\ &= \int_0^{\pi/6} -\frac{a^3}{3} \cos\phi \;|_0^{\pi} \quad d\theta \\ &= \int_0^{\pi/6} \frac{2a^3}{3} \quad d\theta \\ &= \frac{2\pi a^3}{18} = \frac{\pi a^3}{9} \end{aligned}$$

If we had let θ range from 0 all the way to 2π, we would have obtained the volume of the full sphere, which is $\frac{4}{3}\pi a^3$. This observation leads to a quick way to check our answer geometrically. Since $\pi/6$ is one twelfth of 2π, we should expect the wedge to have one twelfth of the volume of the sphere. Indeed,

$$\frac{1}{12} \cdot \frac{4}{3}\pi a^3 = \frac{\pi a^3}{9}$$

as desired.