Math 1272 Calculus II, Fall 2007
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Integration by parts!

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1. The Formula

Unhelpfully, integration by parts helps us compute integrals by turning integrals we can't do into integrals we can. Of course, this describes just about any method used to do anything (rarely do you take something you can do and turn it into something you can't).

More helpfully, integration by parts is the product rule for differentiation plus the fundamental theorem of calculus. By this, I mean say we have two functions u and v. The product rule says that

$$(uv)' = u\ dv + du\ v$$

where du and dv denote the derivatives of u and v (in the obvious order). Then, clearly, the same equality holds if we integrate everything

$$\int (uv)' = \int u\ dv + \int v\ du$$

By the fundamental theorem of calculus (err... the Fundamental Theorem of Calculus),

$$\int (uv)' = uv + C$$

Re-arranging, we have the true formula

$$\int u\ dv = uv - \int v\ du$$

This can also be written

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$$

The point is that sometimes a function is simpler when it is differentiated (for example, x becomes 1), while another function does not become too much worse when integrating (for example, e^x stays the same), so using integration by parts can turn a complicated integral into a simpler one.

Also worth mentioning is integration by parts for definite integrals. The formula here is

$$\int_a^b f(x)g'(x)\ dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\ dx$$

recalling that $f(x)g(x)\Big|_a^b$ means $f(b)g(b) - f(a)g(a)$

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2. ILATE

There is a rule of thumb for deciding, given a product of two functions, which should be differentiated, and which should be integrated. The acronym is ILATE. "I" is for inverse trig functions (arcsin, $\cos^{-1}$, arctan, etc), "L" is for logarithmic functions, "A" is for polynomials (algebraic functions), such as x^4, "T" is for trig functions (sin, cos), and "E" is for exponentials (e^x, 2^x). The idea is that you look at how your functions fall on the ILATE range. The one that comes first is differentiated, and the one that comes second is integrated.

The idea is that I's and L's become much more complicated when integrated, but become simpler when differentiated; A's become slightly more complicated when integrated and slightly simpler when differentiated; T's and E's stay about the same either way.

Finally, note what is missing. It is not obvious how to deal with square roots, and some trig functions, like tan and sec, become horrible when integrated, and I can't think of any situations where differentiation would help.

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3. Some examples

As a warm-up, we calculate

$$\int x\cos(x)\ dx$$

By ILATE, we want to differentiate $f(x) = x$ and integrate $g'(x) = \cos(x)$. So $f'(x) = 1$ and $g(x) = \sin(x)$. Thus, IBP gives

$$\int x\cos(x)\ dx = x\sin(x) - \int \sin(x)\ dx = x\sin(x) + \cos(x) + C$$

Slightly more difficult, we calculate

$$\int x^2 e^x\ dx$$

Again, we differentiate $f(x) = x^2$ and integrate $g'(x) = e^x$, so $f'(x) = 2x$ and $g(x) = e^x$. So we get

$$\int x^2 e^x\ dx = x^2 e^x - \int 2xe^2\ dx = x^2 e^x - 2\int xe^x\ dx$$

We have to use integration by parts to calculate that final integral, setting $f(x) = x$ and integrate $g'(x) = e^x$, so $f'(x) = 1$ and $g(x) = e^x$. So we get

$$\int x^2 e^x\ dx = x^2 e^x - 2\int xe^x\ dx = x^2 e^x - 2\Big(xe^x - \int e^x\ dx\Big) = x^2 e^x - 2\Big(xe^x - e^x\Big)$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

I want to briefly mention the "tabular" method for calculating integrals where you need to use integration by parts multiple times. For example, we use want to calculate

$$\int x^4 \sin(x)\ dx$$

We make two columns (I will be using rows since they are easier to typeset). The first has the successive derivatives of x^4 until we get zero, and the second has the successive integrals of $\sin(x)$ until the row has the same length as the first.
1 Derivatives of x^4: x^4, $4x^3$, $12x^2$, $24x$, 24, 0
2 Integrals of $\sin(x)$: $\sin(x)$, $-\cos(x)$, $-\sin(x)$, $\cos(x)$, $\sin(x)$, $-\cos(x)$
Now, match the first entry in the first row with the second in the second row (and the second entry in the first row with the third in the second row), and add them together, **alternating the signs**, so we get

$$\int x^4 \sin(x)\ dx = -x^4\cos(x) + 4x^3\sin(x) + 12x^2\cos(x) - 24x\sin(x) - 24\cos(x) + C$$

It looks much better when done in columns.

We will finish off this sheet with two tricky examples. The first is to calculate

$$\int ln(x)\ dx$$

The trick is to realize that there is secretly a second function here, the constant function 1. So $ln(x)$ is really $1 \cdot ln(x)$. Since 1 is considered an algebraic function, using ILATE, we differentiate $f(x) = ln(x)$ and integrate $g'(x) = 1$, so $f'(x) = \frac{1}{x}$ and $g(x) = x$. So we get

$$\int ln(x)\ dx = x \cdot ln(x) - \int \frac{1}{x} x\ dx = x \cdot ln(x) - \int dx = x \cdot ln(x) - x + C$$

You use the same method for calculating the integrals of inverse trig functions, but the second integral you get is more complicated, but requires only a simple substitution to solve.

The second is to calculate

$$\int e^{3x} \sin(x)\ dx$$

The tricky-ness of this is not obvious at first. We set $f(x) = \sin(x)$ and $g'(x) = e^{3x}$, so $f'(x) = \cos(x)$ and $g(x) = \frac{1}{3}e^{3x}$. IBP gives that the integral is equal to

$$\frac{1}{3}e^{3x} \sin(x) - \frac{1}{3} \int e^{3x} \cos(x)\ dx$$

Since we can't compute the integral in this equation, we might as well use IBP again, with $f(x) = \cos(x)$ and $g'(x) = e^{3x}$, so $f'(x) = -\sin(x)$ and $g(x) = \frac{1}{3}e^{3x}$. Then

$$\int e^{3x} \cos(x)\ dx = \frac{1}{3}e^{3x} \cos(x) + \frac{1}{3} \int e^{3x} \sin(x)\ dx$$

On first seeing this, a person might assume that we have not made any progress. In fact, we have gone full circle. In this case, this is a good thing! What we have found (plugging everything into the equations) is that

$$\int e^{3x} \sin(x)\ dx = \frac{1}{3}e^{3x} \sin(x) - \frac{1}{3}\left(\frac{1}{3}e^{3x} \cos(x) + \frac{1}{3} \int e^{3x} \sin(x)\ dx\right)$$

Simplifying, this becomes

$$\int e^{3x} \sin(x)\ dx = \frac{1}{3}e^{3x} \sin(x) - \frac{1}{9}e^{3x} \cos(x) - \frac{1}{9} \int e^{3x} \sin(x)\ dx$$

Since the integrals are the same, we can move the left-hand-side one over to the right and add them together, giving

$$\frac{10}{9} \int e^{3x} \sin(x)\ dx = \frac{1}{3}e^{3x} \sin(x) - \frac{1}{9}e^{3x} \cos(x)$$

Thus, multiplying both sides by $\frac{9}{10}$,

$$\int e^{3x} \sin(x)\ dx = \frac{3}{10}e^{3x} \sin(x) - \frac{1}{10}e^{3x} \cos(x)$$

So we were able to calculate an integral using integration by parts, even though the integration by parts "process" did not terminate.