Introduction to L-functions
(March 21, 2006)

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This is a note on L-functions designed to motivate more advanced constructions, e.g. L-functions of automorphic forms (whatever they are). Prerequisites vary wildly, but at worst are not too far from general knowledge (hopefully). My lack of historical knowledge should be evident throughout.

1. In the Beginning...

We discuss the simplest examples.

1.1. The Riemann Zeta Function

Our story starts with the Riemann zeta function. The vast majority of the known properties of the zeta function are of no interest to us, but every story needs a beginning.

We define

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

The sum converges for $Re(s) > 1$. The first important property of $\zeta(s)$ is that it is expressible as an *Euler product*

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

We will see later that the defining $\zeta(s)$ as a product is, in a way, more natural.

The other important properties of $\zeta(s)$ (for us) are its *meromorphic continuation* and *functional equation.* In their proper form, neither of these properties apply directly to $\zeta(s)$, but, properly understood, $\zeta(s)$ is not actually what captures our interest: $\zeta(s)$ is, in a sense, incomplete.

We define the completed zeta function to be

$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

We should make the obligatory note that $\Gamma(s)$ is never zero, so the zeros of $\zeta(s)$ are the same as those of $\Lambda(s)$.

We will prove later, by general means, that $\Lambda(s)$ has a meromorphic continutation to all of $\mathbf{C}$ with simple poles at $s = 0$ and $s = 1$ (the residues of which have number theoretic significance) and satisfies the functional equation

$$\Lambda(s) = \Lambda(1 - s).$$

1.2. Dirichlet L-Functions and Dedekind Zeta Functions

The other elementary example of an L-function are defined as follows. Start with a character

$$\chi : (\mathbf{Z}/l\mathbf{Z})^{\times} \longrightarrow \mathbf{C}^{\times}.$$

Pull it back to a multiplicative function on $\mathbf{Z}$ by setting $\chi(m) = 0$ whenever m is not invertible modulo l. Then we define the Dirichlet L-function associated to χ by

$$L(\chi, s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

This converges for $Re(s) > 1$. It also has an Euler product

$$L(\chi, s) = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$

Similarly to the zeta function, the (completion of a) Dirichlet L-function possesses a meromorphic continuation and a functional equation. The completion is obtained by multiplying $L(\chi, s)$ by suitable gamma functions as above. The exact process is not clear from an elementary viewpoint, so we will ignore it for now.

If we call the completed L-function $\Lambda(\chi, s)$, the functional equation is

$$\Lambda(\chi, s) = \epsilon(\chi) l^{1/2-s} \Lambda(\bar{\chi}, 1-s),$$

where l is from the definition of χ and $\epsilon(\chi)$ can be explicitly calculated.

Finally, we just barely mention the definition of the Dedekind zeta function, which requires some algebraic number theory (which we will not touch). Let k be a finite extension of $\mathbf{Q}$. Then the Dedekind zeta function of k is defined to be

$$\zeta_k(s) = \sum_{\mathbf{a}} (N\mathbf{a})^{-s},$$

where $\mathbf{a}$ is an ideal of $\mathcal{O}$, the ring of integers of k (i.e. the integral closure of $\mathbf{Z}$ in k) and $N\mathbf{a}$ is the size of $\mathcal{O}/\mathbf{a}$.

It has an Euler product

$$\zeta_k(s) = \prod_P (1 - (NP)^{-s})^{-1},$$

where the product is over prime ideals of $\mathcal{O}$. In this case, NP is a power of the prime that P lies over (i.e. a prime p of $\mathbf{Z}$ with P dividing p in $\mathcal{O}$).

1.3. Notes and References

See the first section of the first chapter of Bump's book, or the first two talks in the book Introduction to the Langlands Program edited by Bernstein and Gelbart.

Dirichlet L-functions and Dedekind zeta functions are subsumed under Hecke L-functions. We will prove the meromorphic continuation and functional equation in the third section, when we move to the adelic framework.

2. L-functions for Modular Forms

We quickly overview the classical presentation of L-functions for modular forms (on the upper half plane).

2.1. L-functions for Modular Forms for SL(2,**Z**)

Let f be a holomorphic modular form of weight k for $G = SL(2, \mathbf{Z})$. We could also let G be a congruence subgroup, but this adds in some technicalities, so we will ignore that case.

We take "modular of weight k" to mean that

$$f(gz) = (cz + d)^k f(z),$$

for $g \in G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on the upper half plane by linear fractional transformation.

In particular, $f(z+1) = f(z)$ ($g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$) and $f(-1/z) = z^k f(z)$ ($g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$). The first transformation implies that f has a Fourier expansion:

$$f(z) = \sum_n a_n e^{2\pi i n z}.$$

We take "form" to mean that $a_n = 0$ for $n < 0$.
"Holomorphic" has the usual meaning for a function on the upper half plane.
We can define then the (incomplete) L-function associated to f by

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

It converges for $Re(s) > k$.
We complete $L(f,s)$ as

$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s).$$

Then our result is that $\Lambda(f,s)$ has a meromorphic continuation to all of $\mathbf{C}$, where it is entire if $a_0 = 0$ and has simple poles at $s = 0$ and $s = k$ otherwise, and it satisfies the functional equation

$$\Lambda(f,s) = i^k \Lambda(f, k-s).$$

We use the property that

$$\int_0^\infty e^{-2\pi n y} y^s \frac{dy}{y} = (2\pi)^{-s}\Gamma(s) n^{-s}.$$

So, in the realm of convergence,

$$\sum_{n=1}^{\infty} \int_0^\infty a_n e^{-2\pi n y} y^s \frac{dy}{y} = \Lambda(f,s).$$

Thus, naively assuming everything converges,

$$\int_0^\infty (f(iy) - a_0) y^s \frac{dy}{y} = \Lambda(f,s).$$

For simplicity we assume that $a_0 = 0$ (i.e. f is a cusp form). We (theoretically) know that cusp forms have rapid decay as $y \longrightarrow \infty$[1] and as $y \longrightarrow 0$, by the modular property:

$$f(iy) = f(i/y)(iy)^{-k}.$$

So the integral

$$\int_0^\infty f(iy) y^s \frac{dy}{y}$$

makes sense for all $s \in \mathbf{C}$. This shows that our naive assumption is valid and gives an analytic continuation of $\Lambda(f,s)$ to $\mathbf{C}$.
Again using the modular property, we have

$$\int_0^\infty f(iy) y^s \frac{dy}{y} = i^{-k} \int_0^\infty f(i/y) y^{s-k} \frac{dy}{y} = i^k \int_0^\infty f(iy) y^{k-s} \frac{dy}{y}.$$

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[1] In fact, a modular form minus its zeroth Fourier coefficient has rapid decay.

The left hand side is $\Lambda(f, s)$, and the right hand side is $i^k \Lambda(f, k-s)$, whence the functional equation.

2.2. Discussion

The point of the previous section is to show that we prove meromorphic continuation and functional equation of L-functions by writing them as integrals. In fact, the *integral representation* of an L-function is the vantage point from which "most" of its "important" properties are accessible.

On the more philosophical side, let me mention that a person can attach an L-function to an arithmetic object (say an elliptic curve). The way we prove such L-functions have meromorphic continuations and functional equations is to show that they coincide with L-functions of automorphic objects (say a modular form).

One result of Wiles' work (completed by Breuil, Conrad, Diamond, and Taylor) is that L-functions attatched to elliptic curves (called Hasse-Weil L-functions) coincide with L-functions attatched to cusp forms for certain congruence subgroups, hence have the hoped for properites.

An interesting contrast is that for Hecke L-functions, both the arithmetic properties (e.g. properties of primes) and the analytic properties (e.g. meromorphic continuation) are "near the surface," while for more complicated L-functions (e.g. Hasse-Weil, Artin, or automorphic ones) you have either arithmetic information encoded in the definition (in the Hasse-Weil or Artin case) and no tools to prove continuation and functional equation (which seems to be necessary in order to prove arithmetic results), or the analytic information is more readily available (in the automorphic case),[2] but there is no obvious arithmetic information.

The names associated to this idea are Grothendieck's theory of Motives and the Langlands Program. Both of these subjects are very deep and very little is known.

2.3. Notes and References

See the third section of the first chapter of Bump's book for this material. Knapp's book on elliptic curves contains it as well as the adjustment for arbitrary congruence subgroups. The book by Diamond and Shurman is a more recent version of Knapp's book and spends more time on the connections with elliptic curves.

A good question that we have ignored is when the L-function of a modular form has an Euler product. The answer is when it is an eigenfunction for the Hecke operators. See either Bump, Knapp, or Diamond and Shurman for more in the classical context. Hecke operators have a more digestable interpretation in the adelic viewpoint.

If you are interested, there are several surveys about the Langlands program and the book edited by Bernstein and Gelbart is one. They are comparatively easy to find. On the other hand, for information about Motives, there is the conference proceedings of the same name edited by Jannsen, Kleiman, and Serre. Also some of the papers in the Corvallis conference have some information. Possibly more helpful is the short note "What is... a Motive" by Barry Mazur that was in the Notices of the AMS.

3. "Modern" Interpretation

We reconsider the Riemann zeta function from a modern point of view, showing how it can be written as an integral over the adeles. Then we prove the meromorphic continuation and functional equation of it in this framework (we will literally prove the result for Dedekind zeta functions). The techniques are easily adjusted to apply to all Hecke L-functions.

3.1. Remembrance of Things p-Adic (and Adelic)

We have a classification of locally compact fields. They are $\mathbf{R}$, $\mathbf{C}$, finite extensions of $\mathbf{Q}_p$, or finite extensions

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[2] It would be misleading to suggest that proving analytic continuation and functional equation of automorphic L-functions is easy, but it at least seems to be possible.

of $\mathbf{F}_p((t))$. We ignore the last case (the positive characteristic case).

The norm $|\ |_p$ on $\mathbf{Q}_p$ is defined as follows: on $\mathbf{Z}$, define $|n|_p$ to be p^{-r} where p^r is the largest power of p that divides n. We extend this definition to $\mathbf{Q}$ in the obvious way and extend it to $\mathbf{Q}_p$ by continuity (we define $|0|_p = 0$).

Define $\mathbf{Z}_p$ to be the elements of $\mathbf{Q}_p$ with $|x|_p \leq 1$, and $\mathbf{Z}_p^\times$ are those with $|x|_p = 1$. $\mathbf{Z}_p$ is a compact discrete valuation ring with unique maximal ideal $p\mathbf{Z}_p$, which we will sometimes write as p.

We have that

$$\mathbf{Q}_p^\times = \bigcup_{n=-\infty}^{\infty} p^n \mathbf{Z}_p^\times.$$

If k_v is a finite extension of $\mathbf{Q}_p$, then there is a unique way to extend the absolute value $|\ |_p$ to k_v. Call it $|\ |_v$. We let $\mathcal{O}_{\sqsubseteq}$ be the integral closure of $\mathbf{Z}_p$ in k_v. Then $\mathcal{O}$ is a compact discrete valuation ring with maximal ideal P of elements of absolute value < 1. As above,

$$k_v^\times = \bigcup_{n=-\infty}^{\infty} P^n \mathcal{O}^\times.$$

And if π generates P, $|\pi|_v$ is the size of the field $\mathcal{O}/P$, which is a power of p.

To unify notation, we refer to $\mathbf{R}$ as $\mathbf{Q}_\infty$, the completion of $\mathbf{Q}$ at the infinite prime. For a general finite extension k of $\mathbf{Q}$, there is an infinite prime for every embedding $k \longrightarrow \mathbf{R}$ and for every pair of complex conjugate embeddings $k \longrightarrow \mathbf{C}$. For such a prime v, $k_v = \mathbf{R}$ if the image of k is contained in $\mathbf{R}$, and $k_v = \mathbf{C}$ otherwise. When $k_v = \mathbf{R}$, we take $|\ |_v$ to be the normal absolute value, but when $k_v = \mathbf{C}$, $|\ |_v$ is the square of the normal absolute value (i.e. $|x|_v = x\bar{x}$). In contrast to the "finite" primes, there is no $\mathcal{O}_v$.

Being a locally compact topological group, k_v has a Haar measure, which is unique up to a constant multiple. We write it as dx_v. There is also a Haar measure on $k_v^\times$, written as $d^\times x_v$. The relationship between the two is that $d^\times x_v = |x|_v^{-1} dx_v$. We sometimes drop the subscripts. We normalize the Haar measure so that

$$\int_{\mathcal{O}_v^\times} d^\times x_v = 1.$$

For any number field k (i.e. a finite extension of $\mathbf{Q}$), we can define the adeles A_k to be the "restricted direct product" of the k_v with respect to the $\mathcal{O}_v$. This is the subring of the product $\prod_v k_v$, where $(x_v)_v \in A_k$ means $x_v \in \mathcal{O}_v$ for all but finitely many v.

In the case $k = \mathbf{Q}$, this is the subring of

$$\mathbf{R} \times \prod_p \mathbf{Q}_p$$

where $(x_\infty, x_2, x_3, \ldots)$ is subject to $x_p \in \mathbf{Z}_p$ for all but finitely many p.

We use the restricted direct product instead of the full product to ensure that our object is still locally compact. Hence there is an adelic Haar measure, which is the "product" of the local Haar measures.

3.2. Interpreting the "Local Factors"

The Riemann zeta function is the product of $(1 - p^{-s})^{-1}$ over all primes p. We interpret these pieces as integrals.

We have

$$(1 - p^{-s})^{-1} = \sum_{n=0}^{\infty} p^{-ns} = \sum_{n=0}^{\infty} \operatorname{meas}(\mathbf{Z}_p^\times) p^{-ns} = \sum_{n=0}^{\infty} \int_{p^n \mathbf{Z}_p^\times} |x|^s \ d^\times x$$

$$= \int_{\mathbf{Z}_p - \{0\}} |x|^s \ d^\times x = \int_{\mathbf{Q}_p^\times} 1_{\mathbf{Z}_p}(x) |x|^s \ d^\times x,$$

where $1_{\mathbf{Z}_p}(x)$ is the characteristic function of $\mathbf{Z}_p$.

Hence we have

$$\zeta(s) = \prod_p \int_{\mathbf{Q}_p^\times} 1_{\mathbf{Z}_p}(x)|x|^s \; d^\times x.$$

In a similar way, we can interpret the Dedekind zeta function for a finite extension $k/\mathbf{Q}$ as

$$\zeta_k(s) = \prod_P \int_{k_P^\times} 1_{\mathcal{O}_\mathcal{P}}(x)|x|^s \; d^\times x,$$

where $\mathcal{O}$ is the integral closure of $\mathbf{Z}$ in k and the P are primes of $\mathcal{O}$.

3.3. The "Infinite" Primes

The "local factors" at the infinite places of k are as follows.

At real places it is

$$\pi^{-s/2}\Gamma(s/2),$$

and at complex places it is

$$(2\pi)^{1-s}\Gamma(s).$$

Later on we discuss why we know that these are the correct factors.

3.4. The Adelic View

For p finite, let $f_p(x) = 1_{\mathbf{Z}_p}(x)$.

For p real, let $f_p(x) = e^{-\pi x^2}$.

For p complex, let $f_p(x) = e^{-2\pi x\bar{x}}$.

Setting $f(x) = \otimes_p f_p(x)$, we have

$$\Lambda_k(s) = \int_{A^\times} f(x)|x|^s d^\times x.$$

3.5. Proof of Meromorphic Continuation and Functional Equation

We take a slightly more general point of view. Let f be a Schwartz function on A. We then define

$$z(f,s) = \int_{A^\times} f(x)|x|^s d^\times x.$$

The point is that for $Re(s) > 1$, $z(f,s)$ is a holomorphic function taking values in the space of tempered distributions.

Our result is that $z(f,s)$ has a meromorphic continuation to all of $\mathbf{C}$, where is has simple poles at $s = 0$ and $s = 1$. It also satisfies the functional equation

$$z(f,s) = z(\widehat{f}, 1-s).$$

Let us recall the necessary version of Poisson summation:

For $f \in S(A)$ and $t \in A^\times$

$$\sum_{a\in k} f(at) = |t|^{-1} \sum_{a \in k} \widehat{f}(at^{-1}).$$

We split the integral into two parts:

$$z_1(f,s) = \int\limits_{|x|>1} f(x)|x|^s d^\times x,$$

$$z_2(f,s) = \int\limits_{|x|<1} f(x)|x|^s d^\times x.$$

It is an exercise that $A_1^\times$, the ideles of norm one, have zero measure, so can be ignored.
The first integral $z_1(f,s)$ converges for all s. We "wind up" the second integral

$$z_2(f,s) = \sum_{a\in k^\times} \int\limits_{\substack{k^\times\backslash A^\times \\ |x|\le 1}} f(ax)|ax|^s d^\times x$$

$$= \int\limits_{\substack{k^\times\backslash A^\times \\ |x|\le 1}} \left(\sum_{a\in k^\times} f(ax)\right)|x|^s d^\times x$$

$$= \int\limits_{\substack{k^\times\backslash A^\times \\ |x|\le 1}} \left(\sum_{a\in k} f(ax)\right)|x|^s d^\times x - f(0) \int\limits_{\substack{k^\times\backslash A^\times \\ |x|\le 1}} |x|^s d^\times x.$$

Considering the final integral, we write

$$\int\limits_{\substack{k^\times\backslash A^\times \\ |x|\le 1}} |x|^s d^\times x = \int_0^1 \int\limits_{\substack{k^\times\backslash A^\times \\ |x|=t}} |x|^s d^\times x \frac{dt}{t} = \int_0^1 t^{s-1} \int\limits_{\substack{k^\times\backslash A^\times \\ |x|=t}} d^\times x \; dt.$$

Since $k^\times\backslash A_1^\times$ is compact, the integral becomes

$$meas(k^\times\backslash A_1^\times) \int\limits_0^1 t^{s-1} dt = \frac{meas(k^\times\backslash A_1^\times)}{s},$$

where $meas(k^\times\backslash A_1^\times)$ can be computed explicitly:

$$meas(k^\times\backslash A_1^\times) = \frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|D|}w},$$

where r_1, r_2 are the numbers of real and complex places, h is the class number, R the regulator, D the discriminant, and w the number of roots of unity.
We apply the Poisson summation formula to the penultimate integral

$$\int\limits_{\substack{k^\times\backslash A^\times \\ |x|\le 1}} \left(\sum_{a\in k} f(ax)\right)|x|^s d^\times x = \int\limits_{\substack{k^\times\backslash A^\times \\ |x|\le 1}} \left(\sum_{a\in k} \widehat{f}(ax^{-1})\right)|x|^{s-1} d^\times x.$$

We make the change of variables $x \longrightarrow x^{-1}$ to get

$$\int\limits_{\substack{k^\times\backslash A^\times \\ |x|\ge 1}} \left(\sum_{a\in k} \widehat{f}(ax)\right)|x|^{1-s} d^\times x$$

$$= \int\limits_{\substack{k^\times\backslash A^\times \\ |x|\geq 1}} \Bigg(\sum_{a\in k^\times}\widehat{f}(ax)\Bigg)|x|^{1-s}d^\times x + \widehat{f}(0)\int\limits_{\substack{k^\times\backslash A^\times \\ |x|\geq 1}} |x|^{1-s}d^\times x.$$

The final term is subject to the same analysis as above, making it equal to

$$-\widehat{f}(0)\frac{meas(k^\times\backslash A_1^\times)}{1-s}.$$

The first term can be unwound so that

$$\int\limits_{\substack{k^\times\backslash A^\times \\ |x|\geq 1}} \Bigg(\sum_{a\in k^\times}\widehat{f}(ax)\Bigg)|x|^{1-s}d^\times x = \int\limits_{\substack{A^\times \\ |x|\geq 1}} \widehat{f}(x)|x|^{1-s}d^\times x = z_1(1-s,\widehat{f}).$$

In summary, we have

$$z(f,s) = z_1(f,s) + z_1(\widehat{f},1-s) - meas(k^\times\backslash A_1^\times)\Bigg(\frac{f(0)}{s} + \frac{\widehat{f}(0)}{1-s}\Bigg).$$

Since $z_1(f,s)$ is defined for all s, we have the meromorphic continuation; the functional equation follows from the symmetric nature of the equation.

With f defined in the previous section, we have

$$\Lambda_k(s) = z(\widehat{f},1-s),$$

so we need to examine the right hand side.

This is

$$\prod_v \int_{k_v^\times} \widehat{f}_v(x)|x|_v^{1-s}d^\times x_v.$$

For v archimedean, $\widehat{f}_v = f_v$, so nothing changes. For v finite, this equality does not always hold (in fact, it depends on the additive character used to define the Fourier transform), but we still have

$$\int_{k_v^\times} \widehat{f}_v(x)|x|_v^{1-s}d^\times x_v = |d_v|^{1/2-s}(1-q_v^{-s})^{-1},$$

where d_v is the discriminant of $k_v/\mathbf{Q}_v$. Thus we have

$$z(\widehat{f},1-s) = |D_k|^{1/2-s}\Lambda_k(1-s),$$

where D_k is the discriminant of $k/\mathbf{Q}$. So

$$\Lambda_k(s) = |D_k|^{1/2-s}\Lambda_k(1-s).$$

3.6. Hecke L-functions

We will show how the above definitions need to be changed in order to write all Hecke L-functions as integrals. Let ω be a character of $A^\times/k^\times$. Then define

$$z(f,\omega,s) = \int_{A^\times} f(x)\omega(x)|x|^s d^\times x.$$

The above proof works for this integral with very few adjustments.

We need to choose special f_p in order to have the zeta integral factor into the proper Euler product.

3.7. Where Do Local Factors Come From?

When we were looking at the Riemann zeta function, we already knew what the local factors at finite places were, and, since we knew the proper form of its functional equation, we knew what the local factors at the infinite prime should be. We would like to have a more intrinsic definition of the local factors, to guide our search in more exotic locations.

The basic idea is that the local factor should be a "greatest common denominator" among the local zeta integrals as f varies.

3.8. Notes and References

See the first section of the third chapter of Bump's book, the sixth chapter of the book edited by Bernstein and Gelbart, the book by Ramakrishnan and Valenza, or Tate's article in Cassels and Frohlich..

Compare our treatment with the classical treatment of Hecke L-functions as in Lang's Algebraic Number Theory or Neukirch's Algebraic Number Theory.

4. Artin L-functions

We give a sketchy definition of Artin L-functions and discuss their importance.

4.1. Definition

4.2. Properties

It is not hard to prove that Artin L-functions have a meromorphic continuation. An important conjecture is that the continuation is analytic and that there is a functional equation.

When the dimension of the representation is one, class field theory gives that the representation matches a Hecke character, hence the associated Artin L-function coincides with a certain Hecke L-function, which we showed have the desired properties.

Langlands has conjectured that for an n dimensional representation, the associated Artin L-function should coincide with the L-function of a cuspidal automorphic representation of $GL(n)$, which is known to have the desired properties. In this interpretation, we consider Hecke characters as automorphic forms on $GL(1)$.

The fact that the $n = 1$ (abelian) case follows from a seminal result of Twentieth century mathematics (class field theory) should give a hint as to how hard the general case is (I believe the $n = 2$ case is also known, but almost nothing else).