Bernstein Polynomial and D-modules
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This is a paraphrase of chapters 9 and 10 of Coutinho's book A Primer of Algebraic D-modules. The purpose is to prove the existence of the Bernstein polynomial.

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## 1. Basic Facts

Let $k$ be a field of characteristic 0. For some fixed $n$, $k[X]$ denotes the polynomial ring in $n$ variables.

$A_n$ is the subalgebra of $\mathrm{End}_k k[X]$ generated by the operators

$$f \longrightarrow x_i f \quad \text{and} \quad f \longrightarrow \frac{\partial f}{\partial x_i},$$

which will be written as $x_i$ and $\partial_i$.

We have the following commutation relations:

$$[\partial_i, x_j] = \delta_{ij}, \quad [\partial_i, \partial_j] = [x_i, x_j] = 0.$$

From this we can see that $A_n$ has as a basis (as a vector space over $k$), terms of the form $x^\alpha \partial^\beta$.

For $D \in A_n$, the degree of $D$ is the largest $|\alpha| + |\beta|$ occuring in the summands of $D$. The degree has the following properties:

$$\deg(D + D') \leq \max(\mathrm{deg(D)}, \deg(D')),$$

$$\deg(DD') = \deg(D) + \deg(D'),$$

$$\deg[D, D'] \leq \deg(D) + \deg(D') - 2.$$

We define the Bernstein filtration on $A_n$ by making $B_i$ have basis $x^\alpha \partial^\beta$, with $|\alpha| + |\beta| \leq i$. In particular, $B_0 = k$.

For a ring $R$ with filtration $\Gamma$, we construct the associated graded ring

$$gr^\Gamma R = \bigoplus_{i=0}^{\infty} (\Gamma_i / \Gamma_{i-1}).$$

We have that

$$gr^B A_n = S_n,$$

where $S_n$ is isomorphic to a polynomial ring in $2n$ variables,

Let $M$ be a left $A_n$ module with filtration $\Gamma$ with respect to $B$. If $gr^\Gamma M$ is Noetherian over $S_n$, then $M$ is Noetherian over $A_n$.

Let $M$ be a left $A_n$ module with filtration $\Gamma$. Let $N$ be a submodule, and set $\Gamma'$, $\Gamma''$ to be the induced filtrations on $N$, $M/N$. Then we have an exact sequence

$$0 \longrightarrow gr^{\Gamma'} N \longrightarrow gr^\Gamma M \longrightarrow gr^{\Gamma''} M/N \longrightarrow 0.$$

A filtration $\Gamma$ on $M$ is called good if $gr^\Gamma M$ is finitely generated over $S_n$.

Finitely generated modules admit good filtrations: set $\Gamma_i = \sum_j B_i m_j$, where the $m_j$ generate $M$.

Let $\Gamma$, $\Omega$ be two filtrations, if $\Gamma$ is good, then there is a $l$ such that $\Gamma_i \subset \Omega_{i+l}$.

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## 2. Dimension and Holonomy

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### 2.1. Dimension

**(1) Theorem/Definition.** *Let $M = \bigoplus_{i=0}^{\infty}$ be a finitely generated module over $k[X]$. Then there exists a polynomial $\chi(t) \in \mathbf{Q}[t]$, called the* **Hilbert polynomial of M**, *and a positive integer $N$ such that*

$$\chi(s) = \sum_{0}^{s} \dim_k(M_i).$$

Proof: Ommited for now. □

Let $M$ be a finitely generated left $A_n$ module with a good filtration $\Gamma$ with respect to the Bernstein filtration on $A_n$. Denote the Hilbert polynomial of $gr^\Gamma\ M$ over $S_n$ by $\chi(t, \Gamma, M)$.

By the Theorem and the additive nature of dimension of vectorspaces, for $s$ sufficiently large,

$$\chi(s, \Gamma, M) = \sum_{i=0}^{s} \dim_k(\Gamma_i/\Gamma_{i-1}) = \dim_k(\Gamma_s).$$

We call the **dimension of M** $d = d(M)$ the degree of $\chi(t, \Gamma, M)$. The **multiplicity of M** is $m(M) = d!a_d$ where $a_d$ is the leading coefficient of $\chi$. Both are nonnegative integers.

**Example:** $d(A_n) = 2n$. Let $B$ be the Bernstein filtration. We want to find $\chi(t, B, A_n)$, so we want to calculate $\dim_k(B_i)$. Since $B_i$ is spanned by monomials $x^\alpha \partial^\beta$, with $|\alpha| + |\beta| \leq i$, we need to count the nonnegative solutions to

$$\alpha_1 + \ldots + \alpha_n + \beta_1 + \ldots + \beta_n \leq i.$$

There are

$$(i + 2n)(i + 2n - 1) \ldots (i + 1)/(2n)!$$

solutions,[1] so

$$\chi(t, B, A_n) = (t + 2n)(t + 2n - 1) \ldots (t + 1)/(2n)!.$$

This is a polynomial of degree $2n$ with leading coefficient $1/(2n)!$. So $d(A_n) = 2n$ and $m(A_n) = 1$. □

**(2) Theorem.** *Let $M$ be a finitely generated left $A_n$ module, $N$ a submodule of $M$. Then*

$$d(M) = \max(d(N), d(M/N))$$

*and if $d(N) = d(M/N)$, then*

$$m(M) = m(N) + m(M/N).$$

Proof: The finite generation implies that $M$ has a good filtration $\Gamma$. Let $\Gamma'$, $\Gamma''$ be the induced filtrations on $N$, $M/N$. These filtrations are good because $gr^{\Gamma'} N$, $gr^{\Gamma''} M/N$ are finitely generated, being submodules or quotients of the Noetherian $gr^\Gamma M$. So we can define $\chi(t, \Gamma', N)$, $\chi(t, \Gamma'', M/N)$.

From the exact sequence

$$0 \longrightarrow \Gamma'_l/\Gamma'_{l-1} \longrightarrow \Gamma_l/\Gamma_{l-1} \longrightarrow \Gamma''_l/\Gamma''_{l-1} \longrightarrow 0$$

we see that

$$\dim_k(\Gamma'_l/\Gamma'_{l-1}) + \dim_k(\Gamma''_l/\Gamma''_{l-1}) = \dim_k(\Gamma_l/\Gamma_{l-1}).$$

Summing over both sides of this equation for $l = 0, \ldots, s$, when $s$ is sufficiently large, we have

$$\chi(s, \Gamma', N) + \chi(s, \Gamma'', M/N) = \chi(s, \Gamma, M).$$

Since this occurs for all $s$ sufficiently large, we have equality of polynomials

$$\chi(t, \Gamma', N) + \chi(t, \Gamma'', M/N) = \chi(t, \Gamma, M).$$

Since the leading coefficients of these polynomials are nonnegative, the Theorem follows. □

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[1] This is a combinatorics problem, so it will be skipped

**(3) Corollary.** *Let* $M = \bigoplus_{i=1}^{l} M_i$ *be a direct sum of finitely generated left* $A_n$ *modules. Then* $d(M) = \max(d(M_i))$.

**(4) Corollary.** *For a finitely generated module* $M$*,*

$$d(M) \leq 2n$$

Proof: The finite generation gives a surjection

$$A_n^l \xrightarrow{p} M \longrightarrow 0.$$

Then, on one hand, the theorem gives

$$d(A_n^l) = \max(d(M), d(\ker p)).$$

On the other hand the previous corollary gives $d(A_n^l) = d(A_n) = 2n$. So $d(M) \leq 2n$. □

We use the following technical lemma in the proof of Bernstein's Inequailty.

**Lemma.** *Let* $M$ *be a finitely generated left* $A_n$ *module with filtration* $\Gamma$ *with respect to* $B$ *and that* $\Gamma_0 \neq 0$. *Then the* $k$ *linear map*

$$B_i \longrightarrow Hom_k(\Gamma_i, \Gamma_{2i}),$$

*defined by sending* $a \in B_i$ *to the map* $u \longrightarrow au$*, is injective.*

Proof: It is enough to show that $a\Gamma_i \neq 0$ whenever $a \in B_i$ is not 0. We proceed by induction. For $i = 0$, $B_0 = k$, so this follows from the fact that $\Gamma_0 \neq 0$.

Let $a \in B_i$ and assume $a\Gamma_i = 0$. We must have $a \notin k$, so $a$ contains a term of the form $cx^\alpha \partial^\beta$, with $|\alpha| + |\beta| > 0$.

Suppose that some $\alpha_j \neq 0$. Then $[a, \partial_j]$ contains a term $\alpha_j c x^{\alpha - e_j} \partial^\beta$. In particular, it will be a nonzero element of $B_{i-1}$. We have

$$[a, \partial_j]\Gamma_{i-1} = a\partial_j\Gamma_{i-1} + \partial_j a \Gamma_{i-1}.$$

Since $\partial_j \Gamma_{i-1} \subset \Gamma_i$, and $a\Gamma_i = 0$, we have

$$[a, \partial_j]\Gamma_{i-1} = 0,$$

which contradictions induction.

Similarly, suppose that some $\beta_j \neq 0$. Then $[a, x_j]$ contains a term $\beta_j c x^\alpha \partial^{\beta - e_j}$. In particular, it will be a nonzero element of $B_{i-1}$. We have

$$[a, x_j]\Gamma_{i-1} = ax_j\Gamma_{i-1} + x_j a \Gamma_{i-1}.$$

Since $x_j \Gamma_{i-1} \subset \Gamma_i$, and $a\Gamma_i = 0$, we have

$$[a, x_j]\Gamma_{i-1} = 0,$$

which contradictions induction. □

**(5) Theorem (Bernstein's Inequality).** *For a finitely generated module* $M$*,*

$$d(M) \geq n$$

Proof: Create a good filtration $\Gamma$ of $M$ by having a set of generators be in $\Gamma_0$. So $\Gamma_0 \neq 0$. The previous lemma then applies to give an injection

$$B_i \longrightarrow Hom_k(\Gamma_i, \Gamma_{2i}).$$

So $\dim_k(B_i) \leq \dim_k(Hom(\Gamma_i, \Gamma_{2i}))$. But $\dim_k(Hom(\Gamma_i, \Gamma_{2i})) = \dim_k(\Gamma_i)\dim_k(\Gamma_{2i})$.

Writing $\chi(t)$ for $\chi(t, \Gamma, M)$, for $i$ sufficiently large we have that

$$\dim_k(B_i) \leq \chi(i)\chi(2i).$$

But since $d(A_n) = 2n$, we know that $\dim_k(B_i)$ is a polynomial in $i$ of degree $2n$. So

$$2n \leq \deg(\chi(i)\chi(2i)).$$

But $\chi(i)$ is a polynomial of degree $d(M)$, so we have

$$2n \leq 2d(M).$$

The result follows. □

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### 2.2. Holonomy

A finitely generated left $A_n$ module $M$ is **holonomic** if it is zero or if $d(M) = n$.

**(6) Proposition.** *Submodules, quotients, and finite direct sums of holonomic modules are holonomic.*

**(7) Theorem.** *Holonomic modules are Artinian.*

Proof: Let $M$ be holonomic. Suppose it has a descending chain

$$M = N_0 \supset N_1 \supset \ldots \supset N_l.$$

Since submodules and quotients of holonomic modules are holonomic, $n = d(N_i) = d(N_i/N_{i+1})$. Then

$$m(M) = \sum_{i=0}^{l-1} m(N_i/N_{i+1}) + m(N_l).$$

Since multiplicity is a nonnegative integer, this sum must be at least $l$. Hence $M$ cannot have an infinite descending chain. □

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## 3. Examples and the Bernstein Polynomial

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### 3.1. Examples

**(8) Lemma.** *Let $M$ be a left $A_n$ module with filtration $\Gamma$. Suppose there exists $c$ such that for all $j$ sufficiently large,*

$$\dim_k(\Gamma_j) \leq c\frac{j^n}{n!}.$$

*Then $M$ is holonomic with multiplicity $\leq c$. In particular, $M$ is finitely generated.*

Proof: We prove the lemma in two parts. First we show that the conclusion holds for every finitely generated submodule of $M$, and then prove that $M$ itself is finitely generated.

We now show that every finitely generated submodule of $M$ is holonomic with multiplicity $\leq c$. Let $N$ be a finitely generated submodule. Then it has a good filtration $\Omega$. Thus there exists an integer $r$ with $\Omega_j \subset \Gamma_{j+r} \cap N$, so that $\dim_k(\Omega_j) \leq \dim_k(\Gamma_{j+r})$. By our hypothesis, for $j$ sufficiently large,

$$\chi(j, \Omega, N) \leq c(j + r)^n/n!.$$

Hence $d(N) \leq n$ and $m(N) \leq c$. By the Bernstein Inequality, $d(N) = n$. We have finished the first part.

Now we want to prove that $M$ is finitely generated. Choose an ascending chain of finitely generated submodules of $M$

$$N_1 \subset N_2 \subset \ldots \subset N_l.$$

Each are holonomic with $m(N_i) \leq c$. And since the $N_i$ have the same dimension, their multiplicities satisfy $m(N_i) = m(N_{i-1}) + m(N_i/N_{i-1})$. So we have

$$m(N_1) + \sum_{i=2}^{l} m(N_i/N_{i-1}) = m(N_l).$$

Since $m(N_l) \leq c$ and multiplicities are nonegative integers, all ascending chains of finitely generated submodules must have fewer than $c$ steps. This implies that $M$ must be finitely generated.

We can thus apply the first part to $M$ to see that it is holonomic with $m(M) \leq c$. □

**(9) Theorem/Example.** *Let $p \in k[X]$. Then $k[X, p^{-1}]$ is a holonomic $A_n$ module.*

Proof: Let $m$ be the degree of $p$. We define a filtration $\Gamma$ for $k[X, p^{-1}]$, and show that this filtration is amenable to an estimate as in the lemma. Set

$$\Gamma_l = \{f/p^l : \deg(f) \leq (m+1)l\}.$$

It is easy to check that all elements of $k[X, p^{-1}]$ are in some $\Gamma_l$. It is similarly easy to see that

$$B_i \Gamma_l \subset \Gamma_{l+i}$$

and that the dimension of $\Gamma_l$ is bounded by the dimension of the space of polynomials of degree $\leq (m+1)l$, so is finite dimensional. In particular, a similar counting trick as employed in the proof that $d(A_n) = 2n$ shows that

$$\dim_k \Gamma_l \leq ((m+1)l + n) \ldots ((m+1)l + 1)/n!.$$

Since the highest order term in $l$ on the left hand side is $(m+1)^n l^n/n!$, there is a $c$ such that

$$\dim_k \Gamma_l \leq c l^n/n!.$$

We apply the lemma and are done. □

We need one final example to streamline our proof of our main result. Let $s$ be a new variable. We are interested in a certain $A_n(k(s))$ module. Let $p \in k[X]$. We define a symbol $p^s$ and have $\partial_i$ act on it by

$$\partial_i p^s = s p^{-1} \frac{\partial p}{x_i} p^s.$$

So we can define an $A_n(k(s))$ module $k(s)[X, p^{-1}]p^s$, which contains the submodule $A_n(k(s))p^s$. This last module is the important one. Since the proof is the same as for the previous example (with a slight modification to the filtration), we will skip it.

**(10) Theorem/Example.** *Let $p \in k[X]$. Then $k(s)[X, p^{-1}]p^s$ is a holonomic $A_n(k(s))$ module. Hence its submodule $A_n(k(s))p^s$ is also holonomic.*

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## 3.2. The Bernstein Polynomial

We have an automorphism $t$ of $k(s)[X, p^{-1}]p^s$ defined by

$$t(s^i p^s) = (s+1)^i p p^s.$$

This is $A_n(k)$ linear, but not $A_n(k(s))$ linear.

**(11) Theorem.** *Let $p \in k[X]$. Then there exits $B(s) \in k[s]$ and $D(s) \in A_n(k)[s]$ such that*

$$B(s)p^s = D(s)pp^s.$$

Proof: Since $A_n(k(s))p^s$ is holonomic, it is Artinian. Hence the descending chain

$$A_n(k(s))p^s \supset A_n(k(s))pp^s \supset A_n(k(s))p^2p^s \supset \dots$$

terminates. So there exists $k$ such that

$$p^kp^s \in A_n(k(s))p^{k+1}p^s.$$

We apply $t^{-k}$ to both sides[2] , getting

$$p^s \in A_n(k(s))pp^s$$

Clearing denominators of $s$, we get that for some $B(s) \in k[s]$,

$$B(s)p^s \in A_n(k)[s]pp^s.$$

So there is a $D(s) \in A_n(k)[s]$ with

$$B(s)p^s = D(s)pp^s.$$

□

The **Bernstein polynomial of** $p$, written $b_p(s)$, is defined to be the monic generator of the ideal of all possible $B(s)$ satisfying the Theorem.

**Example:** Let $p = x_1^2 + \dots + x_n^2$. Then if $D$ is the differential operator $D = \partial_1^2 + \dots + \partial_n^2$,

$$Dp^{s+1} = 4(s+1)(s+\frac{n}{2})p^2.$$

Then $b_p(s) = (s+1)(s+\frac{n}{2})$.

In general, the Bernstein polynomial is not amenable to computation.

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[2] Note that $t^{-k}(p^kp^s)$ is $p^s$ times a rational function in $s$.