Iwasawa-Tate Theory
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Ben Rosenfield

This will contain my notes on Iwasawa-Tate theory. This will be done first for number fields and simple algebras (a la Weil), giving some number theoretic results. The meat of these notes is a study of the Standard L-functions for GL(n). We extend this to arbitary simple algebras (a la Godement/Jacquet). This is a long term project.

As of now, I have basically completed the first case. There are a number of minor points left implicit or unproven. This should clear up over time.

The necessary representation theory for this part will be included in another note (maybe).

At this point, I am only trying to get the words on paper, once I have enough written down, it will be reorganized.

A is the adeles, J is the ideles. In the local theory, k will denote a local field, while in the global theory, k will be a global field and k_v will denote a completion at the place v.

1. Introduction

2. Summary of Analysis

We will summarize the basic analysis we will encounter.

2.1. The Spaces

We will be interested in Schwartz spaces and spaces of tempered distributions (continuous linear functionals on the Schwartz space) on archimedean, nonarchimedean, and adelic spaces.

In the archimedean case, the Schwartz space S is the space of "rapidly decreasing" C^∞ functions, where "rapidly decreasing" means that

$$\lim_{|x|\longrightarrow\infty} |x^j D^\alpha f(x)| = 0$$

for all integers j and multi-indices α.

An example is the Gaussian function $e^{-\pi|x|^2}$.

In the nonarchimedean case, S is the space of locally constant functions with compact support. In the case where S is defined on k (instead of a space over k), this means that there is an integer n such that $f \in S$ is locally constant on cosets of P^n and has support contained in P^{-n}.

Examples are characteristic functions of compact sets.

In the adelic case, S contains as a dense subspace the space of functions that are finite linear combinations of f such that

$$f(x) = \prod_v f_v(x),$$

where each $f_v \in S(k_v)$ and all but finitely many f_v equal the characteristic function of the ring of integers o_v (such functions that "factor" over the primes will be refered to as factorizable).

We have the right regular representation of $k^\times$ (resp J) on $S(k)$ (resp $S(A)$):

$$r(a)f(x) = f(xa).$$

This extends to a representation on $S'(k)$ (resp $S'(A)$) (the contragredient representation)

$$r'(a)\lambda(f) = \lambda(r(a^{-1}f)).$$

2.2. The Fourier Transform

For the local case, fix $\psi_v : k_v \longrightarrow \mathbf{C}^\times$, a nontrivial additive character. Let $\widehat{k}_v$ be the dual group of k_v. It is an exercise to show that $\hat{k}_v \simeq k_v$ under the isomorphism

$$y \longrightarrow \psi_v(xy).$$

For $f \in S(k_v)$, we define its Fourier transform by

$$\widehat{f}(x) = \int_{k_v} f(y)\psi_v(xy)dy.$$

The map $f \longrightarrow \widehat{f}$ is an isomorphism of $S(k_v)$. There is a unique choice of Haar measure so that

$$f^{\frown\frown}(x) = f(-x).$$

The global case is similar. Fix a nontrivial additive character ψ of A that is trivial on k. Then any character on A/k has the form

$$\psi_a = \psi(ax),$$

for $a \in k$; i.e., the dual group of A/k is k.
Also ψ can be written as a product of local additive characters:

$$\psi = \prod_v \psi_v.$$

For factorizable f, the Fourier transform of f can be define either as the product of the Fourier transforms of the local factors, or, directly, by

$$\widehat{f}(x) = \int_A f(y)\psi(xy)dy.$$

In both the local and global cases, we also define the Fourier transform of a distribution by

$$\widehat{\lambda}(f) = \lambda(\widehat{f}).$$

2.3. Poisson Summation Formula

Poisson Summation. *For $f \in S(A)$ and $t \in J$*

$$\sum_{a\in k} f(at) = |t|^{-1}\sum_{a\in k}\widehat{f}(\frac{a}{t}).$$

Proof: Define a function F on A/k by

$$F(x) = \sum_{a\in k} f((x+a)t).$$

This is a continuous function on a compact group, hence it has a Fourier expansion in terms of the characters of the group:

$$F(x) = \sum_{b\in k} c_b\psi_b(x).$$

We compute the coefficients of the expansion by orthogonality. Set $V = meas(A/k)$.

$$c_b = V^{-1}\int_{A/k} F(x)\psi_b(x)dx = V^{-1}\int_{A/k}\sum_{a\in k} f((x+a)t)\psi(bx)dx$$

$$= V^{-1}\int_{A/k}\sum_{a\in k} f((x+a)t)\psi((x+a)b)dx = V^{-1}\int_A f(xt)\psi(xb)dx$$

$$= (V|t|)^{-1}\int_A f(x)\psi(xbt^{-1})dx = (V|t|)^{-1}\widehat{f}(bt^{-1}).$$

We then have

$$\sum_{a\in k} f(at) = F(0) = \sum_{b\in k} c_b = (V|t|)^{-1}\sum_{b\in k}\widehat{f}(bt^{-1}).$$

We are done if we can prove $V = 1$. We continue in the vein of the previous line:

$$\sum_{a\in k} f(at) = (V|t|)^{-1}\sum_{b\in k}\widehat{f}(bt^{-1}) = V^{-2}\sum_{a\in k} f(at)\hat{}\hat{} = V^{-2}\sum_{a\in k} f(-at) = V^{-2}\sum_{a\in k} f(at).$$

□

Part I: Number fields, GL(1)

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1. Local Zeta Integrals

Let ω be a quasicharacter of the multiplicative group of a local field k, $\omega : k^\times \longrightarrow \mathbf{C}^\times$. Define the quasicharacter ω_s by

$$\omega_s(x) = |x|^s.$$

We will be interested in classifying tempered distributions such that

$$r'(a)\lambda = \omega(a)\lambda.$$

In particular we will show that this space is one-dimensional. We will denote this space by $S'(\omega)$ and call it the space of ω eigendistributions.

The action of $k^\times$ on k by multiplication splits k into two orbits, $\{0\}$ and $k^\times$, so we have an inclusion respecting this action

$$0 \longrightarrow S(k^\times) \longrightarrow S(k) \longrightarrow S(\{0\}) \longrightarrow 0.$$

This gives us an exact sequence via duality

$$0 \longrightarrow S'_0 \longrightarrow S'(k) \longrightarrow S'(k^\times) \longrightarrow 0$$

where S'_0 are distributions supported at $\{0\}$. We take ω eigendistributions to get

$$0 \longrightarrow S'_0(\omega) \longrightarrow S'(\omega) \longrightarrow S'(k^\times)(\omega).$$

Proposition. *$S'(k^\times)(\omega)$ is 1 dimensional, spanned by the distribution (integrate against) $\omega(x)d^\times x$. So for $\lambda \in S'(\omega)$, there is a $c \in \mathbf{C}$ with*

$$\lambda|_{k^\times} = c\cdot\omega(x)d^\times x.$$

Proposition. *For k non Archimedean,*

$$S'_0 = \mathbf{C}\cdot\delta_0 \subset S'(\omega_0),$$

so if $\omega \neq \omega_0$, $S_0'(\omega) = 0$. *For* $k = \mathbf{R}$,

$$S_0' = \bigoplus_{n=0}^{\infty} \mathbf{C} \cdot \frac{d}{dx}^n \delta_0,$$

and $\frac{d}{dx}^n \delta_0 \in S'(x^{-n})$. *For* $k = \mathbf{C}$,

$$S_0' = \bigoplus_{n,m=0}^{\infty} \mathbf{C} \cdot \frac{d}{dz}^n \frac{d}{d\bar{z}}^m \delta_0,$$

and $\frac{d}{dz}^n \frac{d}{d\bar{z}}^m \delta_0 \in S'(z^{-n}\bar{z}^{-m})$.

If ω is unramified, $\omega\omega_s = \omega_0$ for s such that $tq^{-s} = 1$. For such s, $1 \leq \dim S'(\omega\omega_s) \leq 2$. For all other s and for ω ramified, $\dim S'(\omega\omega_s) \leq 1$.

We define local zeta integrals (or zeta distributions) by

$$z(s, \omega, f) = \int_{k^\times} f(x)\omega\omega_s(x)d^\times x = \int_k f(x)\omega(x)|x|^{s-1}\mu dx,$$

for $f \in S(k)$.

Proposition. *For any* f, $z(s, \omega, f)$ *converges for* s *sufficiently large.*

A pleasant calculation shows $r'(a)z(s, \omega) = \omega\omega_s z(s, \omega)$, so $z(s, \omega) \in S'(\omega\omega_s)$.

1.1. Nonarchimedean Theory

Throughout this section, k is a nonarchimedean field with ring of integers o, maximal ideal P, and uniformizer π. Let dx be Haar measure for k, and $d^\times x$ be Haar measure for $k^\times$. So we have $d^\times x = \mu|x|^{-1}dx$, for some $\mu \in \mathbf{C}^\times$.

Let $\omega : k^\times \longrightarrow \mathbf{C}^\times$ be a quasicharacter (a one dimensional representation). We say ω is unramified if it is trivial on the units of o. In this case $\omega(x) = t^{ord(x)}$ for some $t \in \mathbf{C}^\times$ ($t = \omega(\pi)$). So the characters ω_s defined above are unramified. Otherwise a character is called ramified.

If ω is unramified, $\omega\omega_s(x) = |x|^{s+\sigma} = \omega_{s+\sigma}$, where σ is some complex number.

Now we assume ω is unramified, and $\omega(\pi) = t$.

If $f \in S(k)$ is supported away from $\{0\}$, then $z(s, \omega, f)$ converges for all s. Let τ be the element of the group algebra $\mathbf{Z}[k^\times]$ $[1] - [\pi^{-1}]$. So

$$r(\tau)f(x) = f(x) - f(x\pi^{-1}).$$

Then, since f is locally constant, $r(\tau)f$ is supported away from $\{0\}$.

Let $z_0(s, \omega, f)$ denote the distribution

$$\int_{k^\times} r(\tau)f(x)\omega\omega_s(x)d^\times x.$$

Then $z_0(s, \omega)$ converges for all s and $z_0(s, \omega) \in S'(\omega\omega_s)$.

Moreover, let 1_o be the characteristic function of o Then $r(\tau)1_o = 1_o(x) - 1_o(x\pi^{-1})$ is the characteristic funtion of $o^\times$. We have

$$z_0(s, \omega, 1_o) = \int_{o^\times} d^\times x \neq 0.$$

Hence $z_0(s, \omega)$ is never 0,[1] so is a basis vector for $S'(\omega\omega_s)$.

[1] we could normalize the Haar measure so that the above integral is exactly 1

Now assume s is large enough for $z(s,\omega,f)$ to converge. Then

$$z_0(s,\omega,f) = \int_{k^\times} (f(x) - f(x\pi^{-1}))\omega\omega_s(x)d^\times x$$

$$= \int_{k^\times} f(x)\omega\omega_s(s)d^\times x - \omega\omega_s(\pi)\int_{k^\times} f(x)\omega\omega_s(x)d^\times x$$

$$= (1 - tq^{-s})z(s,\omega,f) = L(s,\omega)^{-1}z(s,\omega,f).$$

Therefore we have

$$\frac{z(s,\omega,f)}{L(s,\omega)} = z_0(s,\omega,f).$$

Since the right hand side is entire, so is the left hand side. This provides a meromorphic continuation to $z(s,\omega,f)$ to the entire plane, and shows that it has exactly the same poles as $L(s,\omega)$.

To finish the unramified case, we still need to show that $S'(\omega_0)$ is at most 1 dimensional.

We have the exact sequence

$$0 \longrightarrow \mathbf{C}\cdot\delta_0 \longrightarrow S'(\omega_0) \longrightarrow \mathbf{C}\cdot d^\times x,$$

hence $S'(\omega_0)$ is at least one dimensional. We need to show that the last map is not a surjection (so it is the zero map). This means showing that the preimages of $d^\times x$ in $S'(k)$ are not $k^\times$ invariant.

Let λ be a preimage of $d^\times x$, this means that for $f \in S(k^\times)$,

$$\lambda(f) = \int_{k^\times} f(x)d^\times x.$$

If λ is $k^\times$ invariant, $r'(\pi)\lambda = \lambda$. We calculate that, for $f \in S'(k^\times)$,

$$r'(\pi)\lambda(f) = \int_{k^\times} f(x\pi^{-1})d^\times x = \lambda(f),$$

so they can at most differ by a distribution supported at $\{0\}$. That is,

$$r'(\pi)\lambda = \lambda + c\cdot\delta_0,$$

for some $c \in \mathbf{C}$. We have λ is not $k^\times$ invariant if $c \neq 0$. This requires direct calculation with a suitable Schwartz function.

We have

$$r'(\pi)\lambda(1_0) = \lambda(r(\pi^{-1})1_0).$$

We evaluate

$$r'(\pi)\lambda - \lambda = c\cdot\delta$$

on the characteristic function of o, 1_o. This gives

$$\lambda(1_o(x\pi^{-1}) - 1_o(x)) = c.$$

But

$$1_o(x\pi^{-1}) - 1_o(x) = -1_{o^\times},$$

so $c \neq 0$. Thus λ is not $k^\times$ invariant.

Suppose ω is ramified. We have $k^\times = \bigcup_n \pi^n o^\times$. Since $o^\times$ is compact,

$$\int_{o^\times} \omega(x)d^\times x = 0.$$

If $f \in S(k)$, then f is constant on $\pi^n o^\times$ for n sufficiently large, as is ω_s. Hence for n large enough,

$$\int_{\pi^n o^\times} f(x)\omega\omega_s(x)d^\times x = 0.$$

Hence we have equality

$$z(s,\omega,f) = \int_{k^\times} f(x)\omega\omega_s(x)d^\times x = \int_{k^\times - P^n} f(x)\omega\omega_s(x)d^\times x.$$

In the latter integral, convergence is not a problem for any s (since everything is killed off near zero). This gives an analytic continuation of $z(s,\omega,f)$ to the whole plane.
Let $1_{o^\times}$ be the characteristic function of $o^\times$. Then setting $f = \omega(x)^{-1}1_{o^\times}$, we see that

$$z(s,\omega,f) \neq 0,$$

as in the unramified case.
When ω is ramified, we set $z_0(s,\omega) = z(s,\omega)$ and $L(s,\omega) = 1$, so that the relation

$$z_0(s,\omega) = \frac{z(s,\omega)}{L(s,\omega)}$$

still holds. And we have $z(s,\omega) \in S'(\omega\omega_s)$ for all s.
In case you didn't realize, we have now finished proving that $S(\omega\omega_s)$ is one dimensional in the nonarchimedean case.

1.2. Archimedean Theory

Now, k is either $\mathbf{R}$ or $\mathbf{C}$.
Assume $k = \mathbf{R}$. Then any quasicharacter can be written as $\omega\omega_s$, where $s \in \mathbf{C}$ and $\omega(x) = x^{-n}$, $n = 0, 1$.
Let $L(s,\omega) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$.
Assume $k = \mathbf{C}$. Then any quasicharacter can be written as $\omega\omega_s$, $s \in \mathbf{C}$ and $\omega(z) = z^{-n}\bar{z}^{-m}$, m, $n \in \mathbf{Z}$, one of which is 0.
Let $L(s,\omega) = (2\pi)^{1-s}\Gamma(s)$.
We will show that $z(s,\omega)$ can be meromorphically continued to all $\mathbf{C}$ and that it has the same poles as $L(s,\omega)$, so that

$$z_0(s,\omega) = L(s,\omega)^{-1}z(s,\omega)$$

is entire and nonzero and defines a basis for $S'(\omega\omega_s)$ for all s.

1.3. Functional Equation

Proposition. *If $\lambda \in S'(\omega)$, then $\widehat{\lambda} \in S'(\omega^{-1}\omega_1)$.*

Proof: In preparation, we calculate the Fourier transform of $r(a^{-1})f(x)$. This is

$$(r(a^{-1})f(x))\widehat{\ } = \int_k f(ya^{-1})\psi(xy)dy = |a|\int_k f(y)\psi(xya)dy = |a|r(a)\widehat{f}(x).$$

Therefore,

$$r'(a)\widehat{\lambda}(f) = \widehat{\lambda}(r(a^{-1})f) = \lambda((r(a^{-1})f)\widehat{\ }) = \lambda(|a|r(a)\widehat{f}) = |a|\omega(a)^{-1}\widehat{\lambda}(f).$$

□

Because the $S(\omega)$ are all one dimensional, we have:

Local Functional Equation.

$$z_0(1-s,\omega^{-1})\hat{} = \epsilon(s,\omega,\psi) z_0(s,\omega).$$

This can also be written as

$$\frac{z(1-s,\omega^{-1},\widehat{f})}{L(1-s,\omega^{-1})} = \epsilon(s,\omega,\psi)\frac{z(s,\omega,f)}{L(s,\omega)}.$$

This basically covers the local theory.

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2. Global Zeta Integrals

Let ω be a character of $k^\times\backslash J$, $f \in S(A)$. Define the global zeta integral by

$$z(s,\omega,f) = \int_J f(x)\omega\omega_s(x)d^\times x.$$

If f is factorizable,

$$z(s,\omega,f) = \prod_v z_v(s,\omega_v,f_v).$$

Outside of a finite set of places S, $z_v(s,\omega_v,f_v) = L_v(s,\omega_v)$, so this product converges whenever

$$L^S(s,\omega) = \prod_{v\notin S} L_v(s,\omega_v)$$

converges. Comparing with the Riemann zeta function shows that this occurs for $Re(s) > 1$.
We define the completed L-function

$$\Lambda(s,\omega) = \prod_v L_v(s,\omega_v).$$

We then have the relation

$$z(s,\omega) = \Lambda(s,\omega)z_0(s,\omega),$$

where $z_0(s,\omega) = \otimes_v z_0(s,\omega_v)$.
There is only one major theorem to prove.

Theorem. *The global zeta integral $z(s,\omega)$, (ω unitary) has a meromorphic continuation to all* $\mathbf{C}$ *and satisfies the functional equation*

$$z(1-s,\omega^{-1})\hat{} = z(s,\omega)$$

Proof: We split the integral into two parts:

$$z_1(s,\omega,f) = \int_{|x|>1} f(x)\omega\omega_s(x)d^\times x,$$

$$z_2(s,\omega,f) = \int_{|x|\le 1} f(x)\omega\omega_s(x)d^\times x.$$

The first integral $z_1(s,\omega,f)$ converges for all s. We "wind up" the second integral

$$z_2(s,\omega,f) = \sum_{a\in k^\times} \int_{\substack{k^\times\backslash J\\ |x|\le 1}} f(ax)\omega\omega_s(ax)d^\times x$$

$$= \int_{\substack{k^\times\backslash J\\ |x|\le 1}} \left(\sum_{a\in k^\times} f(ax)\right)\omega\omega_s(x)d^\times x$$

$$= \int\limits_{\substack{k^\times\backslash J\\|x|\le 1}} \Big(\sum_{a\in k} f(ax)\Big)\omega\omega_s(x)d^\times x - f(0)\int\limits_{\substack{k^\times\backslash J\\|x|\le 1}} \omega\omega_s(x)d^\times x.$$

Considering the final integral, we write

$$\int\limits_{\substack{k^\times\backslash J\\|x|\le 1}} \omega\omega_s(x)d^\times x = \int_0^1 \int\limits_{\substack{k^\times\backslash J\\|x|=t}} \omega\omega_s(x)d^\times x\frac{dt}{t} = \int_0^1 t^{s-1}\int\limits_{\substack{k^\times\backslash J\\|x|=t}} \omega(x)d^\times x\ dt.$$

Let J_1 denote the ideles of norm 1. It is well known that $k^\times\backslash J_1$ is compact, hence the inner integral is over a compact set, so unless ω is trivial on J_1, the integral is zero.

If ω is trivial on J_1, it is of the form $|x|^\sigma$, with σ purely imaginary (since ω is unitary). In this case, the integral becomes

$$meas(k^\times\backslash J_1)\int_0^1 t^{s+\sigma-1}dt = \frac{meas(k^\times\backslash J_1)}{s+\sigma},$$

where $meas(k^\times\backslash J_1)$ can be computed explicitly[2] (Tate's theorem 4.3.2 in his thesis) to be

$$meas(k^\times\backslash J_1) = \frac{2^{r_1}(2\pi)^{r_2}hR}{\sqrt{|D|}w},$$

where r_1, r_2 are the numbers of real and complex places, h is the class number, R the regulator, D the discriminant, and w the number of roots of unity.

We apply the Poisson summation formula to the penultimate integral

$$\int\limits_{\substack{k^\times\backslash J\\|x|\le 1}} \Big(\sum_{a\in k} f(ax)\Big)\omega\omega_s(x)d^\times x = \int\limits_{\substack{k^\times\backslash J\\|x|\le 1}} \Big(\sum_{a\in k} \widehat{f}(ax^{-1})\Big)\omega\omega_{s-1}(x)d^\times x.$$

We make the change of variables $x \longrightarrow x^{-1}$ to get

$$\int\limits_{\substack{k^\times\backslash J\\|x|\le 1}} \Big(\sum_{a\in k} \widehat{f}(ax)\Big)\omega^{-1}\omega_{1-s}(x)d^\times x$$

$$= \int\limits_{\substack{k^\times\backslash J\\|x|\ge 1}} \Big(\sum_{a\in k^\times} \widehat{f}(ax)\Big)\omega^{-1}\omega_{1-s}(x)d^\times x + \widehat{f}(0)\int\limits_{\substack{k^\times\backslash J\\|x|\ge 1}} \omega^{-1}\omega_{1-s}d^\times x.$$

The final term is subject to the same analysis as above, making it zero when ω is trivial on J_1 and equal to

$$-\frac{\widehat{f}(0)meas(k^\times\backslash J_1)}{1-s-\sigma}$$

when $|\omega(x)| = |x|^\sigma$.

The first term can be unwound so that

$$\int\limits_{\substack{k^\times\backslash J\\|x|\ge 1}} \Big(\sum_{a\in k^\times} \widehat{f}(ax)\Big)\omega^{-1}\omega_{1-s}(x)d^\times x = \int\limits_{\substack{J\\|x|\ge 1}} \widehat{f}(x)\omega^{-1}\omega_{1-s}(x)d^\times x = z_1(1-s,\omega^{-1},\widehat{f}).$$

[2] not important for our uses

In summary, when ω is nontrivial on J_1, we have

$$z(s,\omega,f) = z_1(s,\omega,f) + z_1(1-s,\omega^{-1},\widehat{f})$$

and when ω is trivial on J_1 (so $\omega = |x|^\sigma$), we have

$$z(s,\omega,f) = z_1(s,\omega,f) + z_1(1-s,\omega^{-1},\widehat{f}) - meas(k^\times\backslash J_1)\left(\frac{f(0)}{s+\sigma} + \frac{\widehat{f}(0)}{1-s-\sigma}\right).$$

As $z_1(s,\omega)$ is defined for all s, we have the desired continuation, along with the fact that $z(s,\omega)$ has simple poles (δ_0 and $\widehat{\delta_0}$) at $-\sigma$ and $1-\sigma$. The functional equation results from the obvious symmetry. □

As a corollary we get

$$z_0(1-s,\omega^{-1})\widehat{\ } = \epsilon(s,\omega) z_0(s,\omega),$$

where

$$\epsilon(s,\omega) = \prod_v \epsilon_v(s,\omega_v,\psi_v)$$

(the fact that the global epsilon factor is independent of the choice of ψ follows from the functional equation of the completed L-function, as the equation is independent of ψ, technically, for now, we should leave it in. But we won't.).

Then we have

$$\Lambda(1-s,\omega^{-1}) z_0(1-s,\omega^{-1})\widehat{\ } = z(1-s,\omega^{-1})\widehat{\ } = z(s,\omega) = \Lambda(s,\omega) z_0(s,\omega).$$

Using the functional equation for $z_0(s,\omega)$ and the fact that it is nowhere zero, we have

$$\Lambda(s,\omega) = \epsilon(s,\omega)\Lambda(1-s,\omega^{-1}).$$

This completes the global theory.

2.1. Number Theoretic Corollary

The following result can be obtained in stronger form via density theorems. But all you really need is the fact that zeta functions[3] have simple poles. I use this result in my notes on class field theory.

Theorem. *Let K/k be a finite extension of degree n. If almost all primes of k split completely over K (meaning there are n primes of K over each prime of k), then $K = k$.*

Proof: Let S be the set of primes of k that do not split completely, and T the set of primes of K lying over those in S. The partial L-functions $L^S(s,1)$ and $L^T(s,1)$ each have a simple pole at $s=1$. If $v \notin S$ and w lies above v, then $q_v = q_w$. As a result of this $L^T(s,1) = L^S(s,1)^n$. Hence $n=1$. □

Part II: Simple Algebras, GL(1) case

This part will basically correspond to sections X.3 and XI.2 of Weil's Basic Number Theory.

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[3] by which I mean L-functions with ω the trivial character

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