The purpose of this note is to give a proof of the fact that finite dimensional topological vectorspaces (over a complete normed division ring) have only one possible topology, as well as related facts, such as a locally compact tvs is necessarily finite dimensional. Many of these facts can be found in Paul Garrett's notes on topological vector spaces, as well as in Rudin's Functional Analysis (where the ground field is assumed complex). The entirety can be found in Bourbaki's TVS (within the first 16 pages). Nothing fancy is used. Most of the major proofs follow Paul Garrett's notes, except for the last one.

The assumptions on the ground field allows us to consider tvs over $\mathbf{R}$, $\mathbf{C}$, as well as (non-Archimedean) local fields (finite extensions of $\mathbf{Q}_p$ or $\mathbf{F}_q((T))$), and Quaternions. In fact, generalizing to this case has no real cost in complexity.

We now recall our basic facts. Let k be a (non-discrete) complete normed division ring. A topological vector space V over k is a k-vector space with a topology such that V is Hausdorff and (vector) addition and scalar multiplication are continuous maps. (One can prove that Hausdorff follows from merely assuming that points are closed)

Note that, as addition is a homeomorphism, many questions about the topology on V come down to questions about the neighborhoods of the origin. For example, a linear function is continuous if and only if it is continuous at the origin.

In what follows, k will always be a non-discrete complete normed division ring and V will be a tvs.

A subset U of V is **balanced** if $xU \subset U$ whenever $|x| \leq 1$. Every open neighborhood U of 0 contains a balanced open neighborhood.

Every neighborhood of the identity is **absorbing**, meaning that for every $v \in V$, there exists $t \in \mathbf{R}$ such that $v \in \alpha U$ for every $\alpha \in k$ with $|\alpha| \geq t$.
To see this, let $v \in V$. Since the map $\alpha \to \alpha v$ is continuous at 0, for any U containing 0 there exists $\epsilon > 0$ such that if $|\alpha| < \epsilon$, then $\alpha v \in U$. But then $v \in \alpha^{-1}U$. Which is what we wanted to show.

We also need the notion of a quotient space. If W is a closed subspace of V, the quotient space V/W is a tvs (since W is closed, V/W is Hausdorff, and since W is a subspace, V/W is a vector space).

The proof of the following result requires some knowledge of nets in a topological space. In the case that the tvs is separable, we can replace the nets by sequences.

Lemma. *A complete subspace of a tvs is closed.*

Proof: Assume $W \subset V$, W is complete. Let x be in the closure of W. Let S be the local basis for the topology of V at 0, and partial order it so that $U \geq U'$ if and only if $U \subset U'$. For each U, choose

$$y_U \in (x + U) \cap W.$$

The net $\{y_U\}$ converges to x, so it is Cauchy. And since W is complete, it must converge in W. Since it can only have one limit, $x \in W$. This finishes the proof.

Proposition. *If V is one dimensional over k (with generator v), then the map $k \to V$ given by $x \to xv$ is a homeomorphism.*

Proof: As scalar multiplication is continuous, we only need to show that the inverse map $xv \to x$ is continuous, and we only need to show that it is continuous at 0. Given $\epsilon > 0$, there exists $x_0 \in k$ with $0 < |x_0| < \epsilon$ (k is non-discrete!). Since V is Hausdorff, we can find a neighborhood U of 0 with $x_0 \notin U$. Replacing U with a smaller neighborhood, we may assume U is balanced. Choose $x \in k$ such that $xv \in U$. If $|x| \geq |x_0|$, then $|x_0 x^{-1}| \leq 1$. Then, as U is balanced,

$$x_0 v = (x_0 x^{-1})(xv) \in U.$$

Contradiction. So if $xv \in U$, then $|x| < |x_0| < \epsilon$. This finishes the proof.

Corollary. *A one dimensional subspace of a tvs is closed.*

Proof: By the above, a one dimensional subspace is homeomorphic to k, which is assumed complete. Hence it itself is complete and so closed by the Lemma.

Theorem. *An n-dimensional tvs V over k is homeomorphic to k^n, moreover, a finite dimensional subspace of an arbitrary tvs is closed.*

Proof: Let $v_1, \dots, v_n$ be a basis of V, we will show the map

$$k \times \dots \times k \to V$$

defined by

$$(x_1, \dots, x_n) \to x_1 v_1 + \dots x_n v_n$$

is a homeomorphism. Continuity comes for free as this map uses only addition and scalar multiplication. We construct a continuous inverse to this map.

We induct on the dimension n of V, the case $n = 1$ just treated. Let $n > 1$ and define

$$W = \sum_{i=1}^{n-1} k v_i.$$

By induction, W is closed. Let q be the quotient map $q : V \to V/W$. We have V/W is one dimensional with basis $q(v_n)$. Therefore we have a homeomorphism $\phi : V/W \to k$ defined by $\phi(xq(v_n)) = x$.

We also have that kv_n is a closed subspace of V, hence we can form the quotient V/kv_n with quotient map q'. The quotient has basis $q'(v_1), \dots q'(v_{n-1})$, and by induction we have a homeomorphism ϕ' defined by

$$x_1 q'(v_1) + \dots x_{n-1} q'(v_{n-1}) \to (x_1, \dots, x_{n-1}).$$

Hence the map

$$(\phi \circ q) \times (\phi' \circ q') : W \oplus kv_n = V \to k^{n-1} \times k \simeq k^n$$

is continuous. It is the inverse of the map created above, and so that map is a homeomorphism.

As for the second statement, k^n is complete, hence so is a finite dimensional subspace homeomorphic to it. By the Lemma, that subspace is closed. This finishes the proof.

An example of when this fails: $\mathbf{Q}(\sqrt{2})$ is a vector space over $\mathbf{Q}$. It is a tvs with respect to both the product topology and the induced topology from $\mathbf{R}$. Hence completeness is necessary in the previous theorem.

Theorem. *Let V be a tvs over k. If V is locally compact, then so is k, and V is finite dimensional over k.*

Proof: Let U be a compact neighborhood of 0 in V. Choose $\alpha \in k$ such that $0 < |\alpha| < 1$. Choose finitely many a_i in V such that

$$U \subset \bigcup (a_i + \alpha U).$$

Let W be the subspace spanned by the a_i. It is finite dimensional, hence closed. We will show that $V/W = 0$. First we show V/W is compact.

Let q be the quotient map. Then $q(U) = U'$ is compact. We have

$$U \subset \bigcup (a_i + \alpha U) = \alpha \bigcup (\alpha^{-1} a_i + U) \subset \alpha(U + W) = \alpha U + W.$$

This shows that $U' \subset \alpha U'$ ($U' = U + W$). So $\alpha^{-1} U' \subset U'$ and so $\alpha^{-n} U' \subset U'$ for every n. Since U' is absorbing, $U' = V/W$, and so V/W is compact.

It remains to show that the only compact tvs V is $\{0\}$. We know that V contains a homeomorphic image of k. The norm map is continuous, hence if V is compact, the image of k under the norm is bounded. If V is nonzero, there exists x with nonzero norm, and because k is non-discrete, this norm can be taken to be < 1. So $|x^{-1}| > 1$, and the set $|x^{-n}|$ is unbounded. Contradiction.

Thus V is finite dimensional and locally compact, hence homeomorphic to k^n. It follows that k itself must be locally compact. This completes the proof.

Note: for finite dimensionality, all that is necessary is that V has a neighborhood of 0 with compact closure.

An example of a non-locally compact field: Let k be a non-discrete complete normed field. Then $k((T))$ is not locally compact. As a vector space over k, it is spanned by $1, T, T^2, \ldots$, so it is not finite dimensional, hence not locally compact by the previous result.