Math 8602: REAL ANALYSIS. Spring 2016

Homework #2 (due on Wednesday, February 17).
40 points are divided between 4 problems, 10 points each.

#1. Let a functions $f \in BV([a,b])$ for every subinterval $[a,b] \subset (0,1)$, and its variation on $[a,b]$ does not exceed a constant $C_0 < \infty$ which does not depend on a, b. Show that there exists

$$\lim_{a \searrow 0} f(a).$$

#2. Show that for all $\alpha > 1$ the functions

$$f_\alpha(x) := \sum_{k=1}^{\infty} \frac{\sin\left(2^k x\right)}{2^{k\alpha}} \in BV([0,\pi]),$$

i.e. they have bounded variation on $[0,\pi]$.

#3. Let constants $\alpha, \beta \in (0,1)$ with $\alpha + \beta > 1$, and let functions f, g satisfy

$$[f]_\alpha := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty, \quad [g]_\beta := \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x-y|^\beta} < \infty, \quad \text{and} \quad f, g \equiv 0 \quad \text{on} \quad \mathbb{R}^n \setminus (0,1).$$

Show that there exists

$$\lim_{n \to \infty} S_n, \quad \text{where} \quad S_n := \sum_{j=1}^{2^n} f(2^{-n}j)\big[g(2^{-n}j) - g(2^{-n}(j-1))\big].$$

Hint. Consider $S_{n+1} - S_n$.

#4 (Jensen's Inequality, #42d, p.109). Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$, and let g be a function in $L^1(\mu)$. Show that for any convex function F on $\mathbb{R}^1$, we have

$$F\left(\int_X g\, d\mu\right) \leq \int_X F(g)\, d\mu.$$

Hint. You can use without prove the fact that any convex function can be represented as an upper bound of linear functions:

$$F(u) = \sup_{\alpha \in A}(k_\alpha u + b_\alpha).$$