

**MATH 2263, Midterm 3, Spring 2005, April 21
Problems and Solutions**

1. Evaluate

$$\iiint_E z \, dV,$$

where the region E in the three dimensional space with coordinates (x, y, z) is described by the inequalities

$$x^2 + y^2 + z^2 \leq \sqrt{x^2 + y^2}, \quad z \geq 0.$$

Solution. In spherical coordinates, we have

$$x^2 + y^2 + z^2 = \rho^2, \quad \sqrt{x^2 + y^2} = \rho \sin \varphi, \quad z = \rho \cos \varphi,$$

so that the inequalities

$$x^2 + y^2 + z^2 \leq \sqrt{x^2 + y^2}, \quad z \geq 0$$

are equivalent to

$$0 \leq \rho \leq \sin \varphi, \quad 0 \leq \varphi \leq \pi/2.$$

Therefore

$$\iiint_E z \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\sin \varphi} \rho \cos \varphi \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{\pi}{2} \int_0^{\pi/2} \sin^5 \varphi \cdot \cos \varphi \, d\varphi = \frac{\pi}{12} [\sin^6 \varphi]_0^{\pi/2} = \frac{\pi}{12}.$$

2. Evaluate the double integral

$$\iint_S e^{x+y} \, dA,$$

where S is the square with vertices $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, and $D(0, -1)$.

Solution. The square D is bounded by lines $x + y = \pm 1$, $x - y = \pm 1$. Choosing

$$u = x - y, \quad v = x + y$$

as new coordinates, we have

$$x = \frac{1}{2}(u + v), \quad y = \frac{1}{2}(-u + v), \quad \text{the Jacobian } \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2},$$

and hence

$$\iint_D e^{x+y} \, dA = \int_{-1}^1 \int_{-1}^1 e^v \cdot \frac{1}{2} \, du \, dv = \int_{-1}^1 e^v \, dv = e - \frac{1}{e}.$$

(over)

3. Use Green's Theorem to evaluate

$$\int_C 2(x^2 + y^2)dx + (x + y)^2dy,$$

where C is the triangle joining the points $(0, 0)$, $(1, 0)$, and $(1, 1)$ in the counterclockwise direction.

Solution. Since C is the boundary of $D : \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$, we have

$$\begin{aligned} \int_C 2(x^2 + y^2)dx + (x + y)^2dy &= \int_D \left[\frac{\partial((x + y)^2)}{\partial x} - \frac{\partial(2(x^2 + y^2))}{\partial y} \right] dA \\ &= \int_0^1 \left[\int_0^x 2(x - y)dy \right] dx = \int_0^1 x^2 dx = \frac{1}{3}. \end{aligned}$$

4. In the three dimensional space with coordinates (x, y, z) , find the surface area of the part of the cylinder $x^2 + z^2 = 1$ that lies inside of another cylinder $x^2 + y^2 = 1$.

Solution. By symmetry, the part of our surface that lies in the octant

$$\{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0\}$$

has $\frac{1}{8}$ of the total surface area A . Moreover, it is represented as the graph of the function

$$z = f(x, y) = \sqrt{1 - x^2}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{1 - x^2}.$$

Therefore,

$$\begin{aligned} \frac{1}{8}A &= \iint_D \sqrt{1 + f_x^2 + f_y^2} dA = \iint_D \sqrt{\frac{x^2}{1 - x^2} + 1} dA \\ &= \iint_D \frac{1}{\sqrt{1 - x^2}} dA = \int_0^1 \left[\int_0^{\sqrt{1 - x^2}} \frac{1}{\sqrt{1 - x^2}} dy \right] dx = 1, \end{aligned}$$

and $A = 8$.