MATH 2263, Spring 2005. Topics and Typical Problems for Midterm 1
Textbook: James Stewart "Multivariable Calculus", 5th.

Sec. 13.1, 13.2: Limits, derivatives and integrals of vector functions. The unit tangent vector, the tangent line to a given curve.
Sec. 13.3: The arc length, the curvature of a curve (no normal and binormal vectors).
Sec. 13.4: Velocity and acceleration (no Kepler's laws).
Sec. 14.2: Limits and continuity.
Sec. 14.3: Partial derivatives.
Sec. 14.4: Tangent planes.
Sec. 14.5: The chain rule. Implicit differentiation.

TYPICAL PROBLEMS

1(a) Given a point $P_0(c, 3c^3 - c + 6)$ on the plane curve $y = 3x^3 - x + 6$, find parametric equations

$$x = x_0 + at, \quad y = y_0 + bt$$

for the tangent line to the curve at point P_0.

1(b) Find the constants p and c for which the tangent line to this curve at point P is given by the equation $y = px$.

2(a) Find the length of the curve

$$\vec{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t \rangle, \quad 0 \le t \le 2\pi.$$

2(b) Find the curvature of $\vec{r}(t)$ for arbitrary t.

3 Given the plane curve $4x^2 + y^2 = 4$, find the maximal and minimal values of its curvature. Find the points on this curve where the curvature is maximal or minimal.
Hint: Use the parametric representation $x = \cos t, y = 2 \sin t$.

4(a) Find the length of the curve defined as the intersection of two surfaces

$$x^2 + y^2 + z^2 = 6 \quad \text{and} \quad y - z = 2.$$

4(b) Find the curvature of this curve at the point $P(2, 1, -1)$.

5. If $z = f(x, y) = x \ln(xy)$,
5(a) (5 points) find the first partial derivatives f_x, f_y;
5(b) (5 points) find the third partial derivative $f_{xxy} = \partial^3 z / \partial^2 x \partial y$.

6. Find

$$\left(\frac{\partial z}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2, \qquad \text{where } z = e^x \cos y, \;\; x = st, \;\; y = \frac{1}{2}(s^2 - t^2).$$

7. Let z be defined implicitly as a function of x and y by the equation

$$x^2 + 2y^2 + 3z^2 + 2xy + 2xz + 4yz = 2. \qquad (*)$$

7(a) Find the first partial derivatives

$$z_x = \frac{\partial z}{\partial x} \quad \text{and} \quad z_y = \frac{\partial z}{\partial y} \qquad \text{at the point} \quad P(1,-1,1).$$

7(b) Differentiating the equality (*) twice with respect to x, find the second partial derivative

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} \qquad \text{at the point} \quad P(1,-1,1).$$

7(c) Find all the points on the surface described by the equation (*), at which the tangent plane to this surface is horizontal, i.e. it has the form $z = c = const$.

SOLUTIONS

1(a). The curve has vector equation $\vec{r}(x) = \langle x, 3x^3 - x + 6\rangle$. Therefore, $\vec{OP_0} = \langle x_0, y_0\rangle = \langle c, c^3 - c + 6\rangle$. The tangent vector at P_0 is $\vec{r}'(x) = \langle a, b\rangle = \langle 1, 9c^2 - 1\rangle$. Hence the parametric equations

$$\begin{aligned} x &= x_0 + at, \quad y = y_0 + bt \\ \text{where} \qquad x_0 &= c,\ y_0 = c^3 - c + 6,\ a = 1,\ b = 9c^2 - 1. \end{aligned}$$

1(b). From the above equation it follows:

$$y = y_0 + bt = (3c^3 - c + 6) + (9c^2 - 1)(x - c).$$

We need to find such c that this equation has the form $y = px$. For $x = 0$, we get $(9c^2 - 1)(-c) + (3c^3 - c + 6) = 0$. Therefore, $c^3 = 1$, $c = 1$, and $p = 9c^2 - 1 = 8$.

2. We have $\vec{r} = \vec{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, 0\rangle$,

$$\begin{aligned} \vec{r}' &= \langle t\cos t, t\sin t, 0\rangle, \quad \vec{r}'' = \langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle; \\ \vec{r}' \times \vec{r}'' &= \langle 0, 0, t^2\rangle, \quad |\vec{r}'| = t, \quad |\vec{r}' \times \vec{r}''| = t^2. \end{aligned}$$

$$\text{The length } L = \int_0^{2\pi} |\vec{r}'(t)|\, dt = \int_0^{2\pi} t\, dt = 2\pi^2, \text{ the curvature } k = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{1}{t}.$$

3. We have $\vec{r} = \langle x, y, 0\rangle = \langle \cos t, 2\sin t, 0\rangle$,

$$\begin{aligned} \vec{r}' &= \langle -\sin t, 2\cos t, 0\rangle, \quad \vec{r}'' = \langle -\cos t, -2\sin t, 0\rangle, \quad \vec{r}' \times \vec{r}'' = \langle 0, 0, 2\rangle, \\ |\vec{r}'| &= \sqrt{\sin^2 t + 4\cos^2 t} = \sqrt{1 + 3\cos^2 t}, \qquad k = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = 2(1 + 3\cos^2 t)^{-3/2}. \end{aligned}$$

The curvature has its maximal value $k_{\max} = 2$ at the points $(0, \pm 2)$ corresponding to $\cos t = 0$.

The curvature has its minimal value $k_{\min} = 2(1+3)^{-3/2} = 1/4$ at the points $(\pm 1, 0)$ corresponding to $\cos t = \pm 1$.

4. The distance from the origin to the plane $\{x - z = 2\}$ is $d = 2/\sqrt{2} = \sqrt{2}$. The intersection of this plane with the sphere $\{x^2 + y^2 + z^2 = 6\}$ of radius $R = \sqrt{6}$ is a circle of radius $r = \sqrt{R^2 - d^2} = \sqrt{6 - 2} = 2$. Hence the length $L = 2\pi r = 4\pi$, and the curvature $k = 1/r = 1/2$.

5. $z = f(x,y) = x\ln(xy) = x\ln x + x\ln y$

$$\Longrightarrow \quad z_x = 1 + \ln x + \ln y, \quad z_y = x/y, \quad z_{xxy} = 0.$$

6. By the Chain Rule, we have

$$\begin{aligned} z_s &= z_x \cdot x_s + z_y \cdot y_s = z_x \cdot t + z_y \cdot s, \quad z_t = z_x \cdot x_t + z_y \cdot y_t = z_x \cdot s - z_y \cdot t \\ \Longrightarrow & \quad (z_s)^2 + (z_t)^2 = (s^2 + t^2)[(z_x)^2 + (z_y)^2] \\ &= (s^2 + t^2)[(e^x \cos y)^2 + (-e^x \sin y)^2] = (s^2 + t^2)e^{2x} = (s^2 + t^2)e^{2st}. \end{aligned}$$

7(a). Differentiating the equality $x^2 + 2y^2 + 2xy + (2x + 4y + 3z)z = 2$ with respect to x and y, we get

$$2x + 2y + 2z + (2x + 4y + 6z)z_x = 0, \quad 2x + 4y + 4z + (2x + 4y + 6z)z_y = 0. \ (**)$$

At the point $P(1,-1,1)$, these equalities imply $z_x = z_y = -1/2$.

7(b). Differentiating the first equality in (**) with respect to x, we get

$$2 + 2z_x + (2 + 6z_x)z_x + (2x + 4y + 6z)z_{xx} = 0.$$

At the point $P(1,-1,1)$, this equality implies $z_{xx} = -3/8$.

7(c). We have $z_x = z_y = 0$ at the points where the tangent plane is horizontal. From the equalities (**) we obtain

$$2x + 2y + 2z = 2x + 4y + 4z = 0 \quad \Longrightarrow \quad y = -z, x = 0.$$

The original equality (*) gives us $z^2 = 2$, and hence there are two points at which the tangent plane is horizontal: $(0, \sqrt{2}, -\sqrt{2})$ and $(0, -\sqrt{2}, \sqrt{2})$.