MATH 3283W. Sequences, Series and Foundations.
Final Exam. May 16, 2009, Fraser H 102

80 points are distributed between 8 problems, 10 points each.
You have 120 minutes (1:30 pm – 3:30 pm) to work on these problems.
No books, no notes, and no calculators.

**1.** Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n.$$

**2.** Find the limit

$$\lim_{n\to\infty} \left( \sqrt[6]{n^6 + 3n^5} - n \right).$$

**3.** Prove that if the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

converges at $x = x_1$ and $|x_2| < |x_1|$, then the power series converges absolutely at $x = x_2$.

**4.** Find the limit

$$\lim_{n\to\infty} n \left[ \left(1 + \frac{1}{n}\right)^n - e \right].$$

**5.** Determine for which $x$ the series

$$\sum_{n=0}^{\infty} 2^n \arctan(4^{-n} x)$$

is convergent. If yes, verify whether of not its sum is continuous in $x$.

**6.** Find a series solution in powers of $x$ of the equation

$$(1 - x^2)y'' - 2xy' + 12y = 0,$$

which satisfies the initial conditions $y(0) = 0,\ y'(0) = 1$.

**7.** Consider the sequence

$$a_1 = 1,\ a_2 = 0, \ \text{ and } \ a_{n+2} = \frac{1}{2}(a_n - a_{n+1}) \ \text{ for } \ n = 1, 2, \ldots.$$

Determine whether this sequence is convergent or divergent. If convergent, find its limit.

**8.** Given that $S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2$, evaluate the series

$$S_1 = \sum_{k=1}^{\infty} \frac{1}{(2k)^2}, \qquad S_2 = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}, \qquad S_3 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}.$$

**Useful facts. 1.** (Weierstrass theorem) If $f(x) = \sum f_n(x)$, where $f_n(x)$ are continuous and satisfy $|f_n(x)| \le c_n$ with $\sum c_n < \infty$ for all $x$ in an open set $D$, then $f(x)$ is continuous in $D$.

**2.** Some power series:

$$(1+x)^a = 1 + \frac{ax}{1!} + \frac{a(a-1)x^2}{2!} + \cdots, \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad \text{for} \quad |x| < 1$$