MATH 3283W. Sequences, Series and Foundations.
Final Exam. May 16, 2009. Problems and Solutions

1. Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n.$$

Solution. The series has the form $\sum a_n x^n$, where

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1+\frac{1}{n}}{2\cdot\left(2+\frac{1}{n}\right)} \to L = \frac{1}{4} \qquad \text{as} \quad n \to \infty.$$

Therefore, the radius of convergence $R = 1/L = 4$, and the series converges for $|x| < 4$. At the boundary of this interval, i.e. for $x = \pm 4$, we have the series $\sum A_n$, where $A_n = \pm 4^n a_n$. We have

$$\frac{|A_{n+1}|}{|A_n|} = \frac{4a_{n+1}}{a_n} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}} > 1, \qquad \text{i.e.} \quad 0 < |A_1| < |A_2| < \cdots < |A_n| < \cdots,$$

and A_n does not converge to 0. Hence the series $\sum A_n$ does not converge, and the interval of convergence is $(-4, 4)$.

2. Find the limit

$$\lim_{n\to\infty} \left(\sqrt[6]{n^6 + 3n^5} - n \right).$$

Solution. We can write

$$\sqrt[6]{n^6+3n^5} - n = n \cdot \left[\left(1 + \frac{3}{n}\right)^{1/6} - 1 \right] = \frac{(1+3x_n)^{1/6} - 1}{x_n},$$

where $x_n = 1/n \to 0$ as $n \to \infty$. Using L'Hôpital's Rule, we obtain

$$\lim_{n\to\infty} \left(\sqrt[6]{n^6+3n^5} - n \right) = \lim_{x\to 0} \frac{(1+3x)^{1/6} - 1}{x} = \lim_{x\to 0} \frac{(1+3x)^{-5/6} \cdot 3}{6} = \frac{1}{2}.$$

3. Prove that if the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

converges at $x = x_1$ and $|x_2| < |x_1|$, then the power series converges absolutely at $x = x_2$.

Proof. This is Lemma 7.2 i) in the textbook. Since the series $\sum a_n x_1^n$ is convergent, we have $a_n x_1^n \to 0$ as $n \to \infty$, which in turn implies $|a_n x_1^n| \le C = \mathit{const} < \infty$ for all n. Then

$$\sum_{n=0}^{\infty} |a_n x_2^n| = \sum_{n=0}^{\infty} |a_n x_1^n| \cdot \frac{|x_2|^n}{|x_1|^n} \le C \sum_{n=0}^{\infty} \left(\frac{|x_2|}{|x_1|} \right)^n < \infty, \qquad \text{because} \quad 0 \le \frac{|x_2|}{|x_1|} < 1.$$

4. Find the limit

$$\lim_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)^n - e\right].$$

Solution.

$$n\left[\left(1+\frac{1}{n}\right)^n - e\right] = n\left[e^{n\ln\left(1+\frac{1}{n}\right)} - e\right] = ne\cdot\left(e^{\alpha_n}-1\right),$$

where $\alpha_n = n\ln\left(1+\frac{1}{n}\right) - 1 \to 0$ as $n\to\infty$. Since $\left(e^{\alpha_n}-1\right)/\alpha_n \to 1$, we get

$$\lim_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)^n - e\right] = e\lim_{n\to\infty} n\alpha_n\cdot\frac{\left(e^{\alpha_n}-1\right)}{\alpha_n} = e\lim_{n\to\infty} n\alpha_n = e\lim_{n\to\infty}\left[n^2\ln\left(1+\frac{1}{n}\right) - n\right].$$

One can complete the solution using the power series for $\ln(1+x)$. Alternatively, since $x_n = 1/n \to 0$ as $n\to\infty$, we can use L'Ĥôpital's Rule:

$$e\lim_{n\to\infty}\left[n^2\ln\left(1+\frac{1}{n}\right) - n\right] = e\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2} = e\lim_{x\to 0}\frac{(1+x)^{-1}-1}{2x} = -\frac{e}{2}.$$

5. Determine for which x the series

$$\sum_{n=0}^{\infty} 2^n\arctan(4^{-n}x)$$

is convergent. If yes, verify whether of not its sum is continuous in x.

Solution. Since $|\arctan y| \le |y|$ for all y, we have the estimate $|2^n\arctan(4^{-n}x)| \le 2^{-n}|x| \le 2^{-n}A$ for all n and $|x|\le A$ – an arbitrary positive constant. By the Weierstrass theorem [1], the given series converges uniformly, and its sum is continuous on $[-A,A]$. We can take $A>0$ arbitrarily large, therefore, the sum is continuous in x for all $x\in\mathbb{R}^n$.

6. Find a series solution in powers of x of the equation

$$(1-x^2)y'' - 2xy' + 12y = 0,$$

which satisfies the initial conditions $y(0)=0,\ y'(0)=1$.

Solution. Denote $c_n = y^{(n)}(0)$. By our assumptions $c_0 = 0,\ c_1 = 1$. Instead of comparing the coefficients in power series, on can use the following *Leibnitz formula* for the higher order derivative of a product:

$$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}g^{(k)} = f^{(n)}g + nf^{(n-1)}g' + \frac{n(n-1)}{2}f^{(n-2)}g'' + \cdots.$$

Applying this formula to the given equation at the point $x=0$, we obtain

$$c_{n+2} - \frac{n(n-1)}{2}\cdot 2c_n - 2nc_n + 12c_n = 0,$$

i.e. $c_{n+2} = (n^2+n-12)c_n = (n+4)(n-3)c_n$. Hence

$$0 = c_0 = c_2 = c_4 = \cdots, \qquad c_1 = 1, \quad c_3 = -10, \quad c_5 = c_7 = \cdots = 0.$$

The corresponding coefficients in $\sum a_n x^n$ are $a_n = c_n/n!$. The answer is $y(x) = x - \frac{5}{3}x^3$.

7. Consider the sequence

$$a_1 = 1, \; a_2 = 0, \; \text{ and } \; a_{n+2} = \frac{1}{2}(a_n - a_{n+1}) \; \text{ for } \; n = 1, 2, \ldots .$$

Determine whether this sequence is convergent or divergent. If convergent, find its limit.

Solution. The sequence $a_n = r^n$ satisfies the given equalities (except for initial conditions) if $r^2 = \frac{1}{2}(1-r)$. This quadratic equation has two roots $r_1 = \frac{1}{2}$ and $r_2 = -1$. We can find the desired sequence in the form $a_n = \left(\frac{1}{2}\right)^n C_1 + (-1)^n C_2$. From the initial conditions we get

$$\begin{aligned} a_1 &= \frac{C_1}{2} - C_2 = 1, \qquad a_2 = \frac{C_1}{4} + C_2 = 0, \qquad \text{so that } \; C_1 = \frac{4}{3}, \quad C_2 = -\frac{1}{3}, \\ a_n &= \frac{4 \cdot 2^{-n} - (-1)^n}{3}. \end{aligned}$$

Since $2^{-n} \to 0$ and $(-1)^n$ does not have a limit, the sequence a_n is divergent.

8. Given that $S = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2$, evaluate the series

$$S_1 = \sum_{k=1}^{\infty} \frac{1}{(2k)^2}, \qquad S_2 = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}, \qquad S_3 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}.$$

Solution. It is easy to see that

$$S_1 = \frac{1}{4}S = \frac{1}{24}\pi^2, \qquad S_2 = S - S_1 = \frac{1}{8}\pi^2, \qquad S_3 = S_2 - S_1 = \frac{1}{12}\pi^2,$$

[1] **Weierstrass theorem.** If $f(x) = \sum f_n(x)$, where $f_n(x)$ are continuous and satisfy $|f_n(x)| \le c_n$ with $\sum c_n < \infty$ for all x in an open set D, then $f(x)$ is continuous in D.