MATH 3283W. Sequences, Series, and Foundations: Writing Intensive. Spring 2009

Homework 1 (due on February 12)

I. Writing Intensive Part (preliminary draft is due on February 5)

1 (5 points). Check whether or not each of the following statements can be true for some values ("true" or "false") of P and Q. Write out the truth table for each of these statements.

$$\begin{aligned} A = (P &\implies Q)\&(P \implies \neg Q), \\ B = (P &\implies Q)\&(\neg P \implies Q), \\ C = (P &\implies Q)\&(\neg P \implies \neg Q). \end{aligned}$$

2 (5 points). Prove that a subset of $\mathbb{N}$ is bounded if and only if it is finite.

3 (5 points). Prove that the equation $x^{10} + x + 1 = 0$ does not have rational roots.

4 (5 points). Using the method of mathematical induction, prove that

$$(1+\alpha)^n \geq 1 + n\alpha \quad \text{for all} \quad \alpha \geq -1 \quad \text{and} \quad n \in \mathbb{N}.$$

II. General Part

5 (4 points). Prove the equality (*) in Exercise 4.22 by differentiating the polynomial $1 + x + x^2 + \cdots + x^n$.

6 (5 points). Show that the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is non-decreasing. *Hint.* Write

$$\frac{a_n}{a_{n-1}} = \left(\frac{n^2-1}{n^2}\right)^n \cdot \frac{n}{n-1},$$

and then use Problem 4 in Part I.

For extra credit (2 points): Show that the sequence $b_n = \left(1 + \frac{1}{n}\right)^{n+1}$ is non-increasing.

7 (6 points). Starting from $a_1 = 1$, define the sequence

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \text{for} \quad n \in \mathbb{N}.$$

a) Using induction, show that $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$.

b) Show that $\varepsilon_n = a_n - \sqrt{2} \to 0$ as $n \to \infty$.