MATH 3283W. Sequences, Series, and Foundations: Writing Intensive. Spring 2009

Homework 2 (due on February 26)

I. Writing Intensive Part (preliminary draft is due on February 19)

1 (5 points). Let $f(x)$ be a continuous function on the segment $[0,1]$, such that $f(0) < 0 < f(1)$. Show that $f(c) = 0$ for some $c \in (0,1)$.

2 (7 points). By one of equivalent definitions, a set $K \subset \mathbb{R}^1$ is *closed* is for any convergent sequence $\{s_n\}$, from $s_n \in K$ for all n it follows $L = \lim s_n \in K$. Let $K_1 \supset K_2 \supset K_3 \supset \dots$ be a sequence of bounded, nonempty, closed subsets in $\mathbb{R}^1$. Show that the intersection of all K_j is nonempty, i.e. there is a point $x \in K_j$ for all j.

3 (8 points). This is an extension of Exercise 2.18 on p.13. Consider a function

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where each of $a_0, \dots, a_n$ is an integer (some of them may be non-positive), and $a_n \geq 1$.

(a) Show that there is $k_0 \in \mathbb{N}$ such that $p(k) \geq 2$ for each $k \geq k_0$.

(b) Show that there is an integer $k \geq k_0$ such that $p(k)$ is not a prime number. *Hint.* Consider a polynomial $q(y) = p(k_0 + y)$.

II. General Part

4 (4 points). Exercise 5.6 b) and i). Find

$$\lim_{n\to\infty} n^{(1/\ln n)}, \qquad \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n.$$

5 (3 points). Using the formula $a - b = (a^2 - b^2)/(a+b)$, find

$$\lim_{n\to\infty} \left(\sqrt{n^4 + n^3} + \sqrt{n^4 - n^3} - 2n^2\right).$$

6 (3 points). Let $\{s_n\}$ be a convergent sequence with $\lim s_n = L$. Show that

$$\lim_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{n} = L.$$

7 (5 points). Let $a_1 = 1$, and $a_{n+1} = (1 + a_n)^{-1}$ for $n = 1, 2, 3, \dots$. Find $\lim a_n$. *Hint.* This sequence is NOT monotonic. Consider the differences between a_n and roots of the equation $x = (1+x)^{-1}$.