MATH 3283W. Sequences, Series, and Foundations: Writing Intensive. Spring 2009

Homework 3 (due on April 2)

I. Writing Intensive Part (preliminary draft is due on February 26)

1 (8 points). Let $f(x)$ be a continuous function on the segment $[0,1]$. Show that f is *uniformly continuous* on $[0,1]$, which means: $\forall\varepsilon > 0$, $\exists\delta > 0$, such that from $x, y \in [0,1]$ and $|x-y| < \delta$ it follows $|f(x)-f(y)| < \varepsilon$.

2 (6 points). From Problem 6 in Homework 1 it follows $a_n = \left(1+\frac{1}{n}\right)^n < e < b_n = \left(1+\frac{1}{n}\right)^{n+1}$ for all $n \in \mathbb{N}$. Determine for which $n \in \mathbb{N}$ we have

$$c_n = \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} > e.$$

3 (6 points). Verify whether or not the sequence

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

is convergent.

II. General Part

4 (5 points). For $\lambda = const > 0$, evaluate the improper integrals

$$A(\lambda) = \int_0^\infty e^{-\lambda x}\cos x \, dx, \quad B(\lambda) = \int_0^\infty e^{-\lambda x}\sin x \, dx.$$

Hint. Here the Euler's formula $e^{ix} = \cos x + i\sin x$ is helpful ($i = \sqrt{-1}$).

5 (5 points). Check whether or not the series

$$\sum_{n=0}^\infty \ln\left(1+2^{-2^n}\right)$$

converges. If yes, find its sum.

6 (5 points). Let $A_1 \geq A_2 \geq \cdots \geq A_n \geq \cdots \geq 0$. Show that the series $\sum\limits_{n=1}^\infty A_n$ converges if and only if the series $\sum\limits_{n=1}^\infty B_n$ converges, where $B_n = 2^n A_{2^n}$, i.e. $B_1 = 2A_2,\ B_2 = 4A_4$, etc.

Remark. Problems 5 and 6 are based on the *comparison test*, Theorem 4.1 on p. 103.