MATH 3283W. Sequences, Series, and Foundations: Writing Intensive. Spring 2009

Homework 4 (due on Thursday, April 30)
There is **no writing intensive part in this assignment.**

1. (5 points) Show that the number e is irrational.

Hint. One can follow the steps in Exercise 9.10 on p. 145–146 in the textbook. Equivalently, one can write

$$e = S_n + T_n, \qquad \text{where} \qquad S_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad T_n = \sum_{k=n+1}^{\infty} \frac{1}{k!},$$

for each natural n. If we assume that e is rational, then $n! \cdot e \in \mathbb{N}$ for large $n \in \mathbb{N}$. We also have $n! \cdot S_n \in \mathbb{N}$. Therefore, we must have $n! \cdot T_n = n! \cdot e - n! \cdot S_n \in \mathbb{N}$. In order to get the desired contradiction, one needs to show that $n! \cdot T_n$ cannot be integer.

2. (8 points) Let numbers $r \in \mathbb{N}$ and $p \in (0, 1)$ be given. Define

$$p_k = \binom{k-1}{r-1} p^r q^{k-r} \qquad \text{for} \quad k = r, r+1, r+2, \dots, \qquad \text{where} \quad q = 1 - p.$$

Show that

$$\sum_{k=r}^{\infty} p_k = 1, \qquad \text{and evaluate} \quad \mu_1 = \sum_{k=r}^{\infty} k p_k \quad \text{and} \quad \mu_2 = \sum_{k=r}^{\infty} k^2 p_k.$$

Hint. First verify that

$$p_k = \binom{-r}{k-r} p^r (-q)^{k-r}, \quad \text{where} \quad \binom{a}{j} = \frac{a(a-1)(a-2) \cdot \ldots \cdot (a-j+1)}{j!}.$$

Note that this notation makes sense for any **real** a. Then evaluate ($k = r + j$)

$$\phi(t) = \sum_{k=r}^{\infty} p_k t^k = (pt)^r \sum_{j=0}^{\infty} \binom{-r}{j} (-qt)^j.$$

Finally, find μ_1 and μ_2 by differentiation of $\phi(t)$ at $t = 1$.

3. (7 points) Define

$$p_k(x) = e^{-x} \cdot \frac{x^k}{k!} \qquad \text{for} \qquad k = 0, 1, 2, \dots \qquad \text{and} \quad x \in \mathbb{R}.$$

Find an analytic expression for the function

$$f(x) = \sum_{k=0}^{\infty} \frac{p_k(x)}{k+1}.$$

Moreover, write $f(x)$ as a power series in x and determine its radius of convergence.