

MATH 3283W. Sequences, Series, and Foundations:
Writing Intensive. Spring 2009

Homework 1. Problems and Solutions

I. Writing Intensive Part

1 (5 points). Check whether or not each of the following statements can be true for some values ("true" or "false") of P and Q . Write out the truth table for each of these statements.

$$\begin{aligned}A &= (P \implies Q) \& (P \implies \neg Q), \\B &= (P \implies Q) \& (\neg P \implies Q), \\C &= (P \implies Q) \& (\neg P \implies \neg Q).\end{aligned}$$

Solution. One can compose the truth tables for A, B, C by direct consideration of four possible cases: $(P, Q) = (T, T), (T, F), (F, T)$, and (F, F) . Here the equality $(P \implies Q) = (\neg P \vee Q)$ may be helpful. Some simplifications are possible:

(i) If $P = T$, then one of implications $P \implies Q$ or $P \implies \neg Q$ is false, because either Q or $\neg Q$ is false. Therefore, in this case $A = F$. If $P = F$, then both implications are true (footnote 7 on p.2 in the textbook), so that $A = T$. This means that we always have $A = \neg P$.

(ii) If $Q = T$, then both implications are true, and $B = T$. If $Q = F$, then one of implications is false, and $B = F$. In any case, $B = Q$.

(iii) Since $(\neg P \implies \neg Q)$ is equivalent to $(Q \implies P)$ (p.5 in the textbook), we have $C = (P \iff Q)$.

2 (5 points). Prove that a subset of \mathbb{N} is bounded if and only if it is finite.

Solution. Let A be a subset of \mathbb{N} . The statement $P = (A \text{ is bounded}) = (\exists \text{ a constant } r > 0 \text{ such that } \forall a \in A \text{ we have } |a| \leq r)$. The statement $Q = (A \text{ is finite}) = (A \text{ contains exactly } n \text{ elements for some } n \in \mathbb{N})$. If P is true with a constant $r > 0$, then the number of elements in A cannot exceed r (it is important here that we deal with *natural* numbers), hence Q is true. If Q is true with a constant $n \in \mathbb{N}$, then we can arrange the elements of A in an increasing order: $1 \leq a_1 < a_2 < \dots < a_n$, and P is obviously true with $r = a_n$. Equivalently, one can define r as the maximal element in A (we can do so because A is finite).

3 (5 points). Prove that the equation $x^{10} + x + 1 = 0$ does not have rational roots.

Solution. We follow the proof of the Rational Root Theorem, in a particular case. Suppose that there is a rational root $x = m/n$ of this equation, where m and n are integers, $n \neq 0$. Having in mind possible cancelations, we can assume that m and n do not have nontrivial (bigger than 1) common factors. We can rewrite our equation $(m/n)^{10} + m/n + 1 = 0$ in the form $m^{10} + m \cdot n^9 + n^{10} = 0$. Then $m^{10}/n = -(m \cdot n^8 + n^9)$ is an integer. Therefore, each prime factor $p \geq 2$ of n must be a factor of m . By our assumption, there are no such factors, which is only possible in the case $n = \pm 1$. A similar argument with $n^{10}/m = -(m^9 + n^9) \in \mathbb{Z}$ implies $m = \pm 1$. However, the corresponding values $x = m/n = \pm 1$ do not satisfy the given equation, so that it has no rational roots.

Remark. Some students have noticed that $f(x) = x^{10} + x + 1 > 0$ for all real x , so that the given polynomial does not have *real* roots. This is obvious for $x \geq -1$. For $x < -1$, one can either use fact that $f'(x) = 10x^9 + 1 < 0$ in $(-\infty, -1)$, which implies $f(x) > f(-1) = 1 > 0$, or write $f(x) = (x^5 + 1)^2 - 2x^5 + x > 0$.

4 (5 points). Using the method of mathematical induction, prove that

$$(1 + \alpha)^n \geq 1 + n\alpha \text{ for all } \alpha \geq -1 \text{ and } n \in \mathbb{N}.$$

Solution. Let $P(n)$ denote the given statement. Then $P(1)$ is trivial. Suppose $P(n)$ it is true for some $n \in \mathbb{N}$. Multiplying both sides of $(1 + \alpha)^n \geq 1 + n\alpha$ by $1 + \alpha \geq 0$ (here we need $\alpha \geq -1$), we obtain

$$(1 + \alpha)^{n+1} \geq (1 + n\alpha)(1 + \alpha) = 1 + (n + 1)\alpha + n\alpha^2 \geq 1 + (n + 1)\alpha,$$

i.e. $P(n + 1)$ is true. By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

II. General Part

5 (4 points). Prove the equality (*) in Exercise 4.22 by differentiating the polynomial $1 + x + x^2 + \dots + x^n$.

Solution. We have

$$\begin{aligned} 1 + 2x + 3x^2 + \dots + nx^{n-1} &= \frac{d}{dx}(1 + x + x^2 + \dots + x^n) \\ &= \frac{d}{dx} \left(\frac{x^{n+1} - 1}{x - 1} \right) = \frac{nx^{n+1} - (n + 1)x^n + 1}{(x - 1)^2}. \end{aligned}$$

6 (5 points). Show that the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is non-decreasing.
Hint. Write

$$\frac{a_n}{a_{n-1}} = \left(\frac{n^2 - 1}{n^2}\right)^n \cdot \frac{n}{n-1},$$

and then use Problem 4 in Part I.

For extra credit (2 points): Show that the sequence $b_n = \left(1 + \frac{1}{n}\right)^{n+1}$ is non-increasing.

Solution. Using Problem 4 with $\alpha = -1/n$, we have

$$\frac{a_n}{a_{n-1}} = \frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^{n-1}} = \left(1 - \frac{1}{n^2}\right)^n \cdot \frac{n}{n-1} \geq \left(1 - \frac{n}{n^2}\right) \cdot \frac{n}{n-1} = 1,$$

so that $\{a_n\}$ is non-decreasing. Similarly,

$$\frac{b_{n-1}}{b_n} = \frac{\left(\frac{n}{n-1}\right)^n}{\left(\frac{n+1}{n}\right)^{n+1}} = \left(1 + \frac{1}{n^2 - 1}\right)^{n+1} \cdot \frac{n-1}{n} \geq \left(1 + \frac{n+1}{n^2 - 1}\right) \cdot \frac{n-1}{n} = 1,$$

so that $\{a_n\}$ is non-increasing.

7 (6 points). Starting from $a_1 = 1$, define the sequence

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right) \quad \text{for } n \in \mathbb{N}.$$

a) Using induction, show that $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$.

b) Show that $\varepsilon_n = a_n - \sqrt{2} \rightarrow 0$ as $n \rightarrow \infty$.

Solution. a) The function $f(x) = \frac{1}{2}(x + 2x^{-1})$ has derivative $f'(x) = \frac{1}{2}(1 - 2x^{-2})$ in the interval $(1, 2)$, which changes sign from "-" to "+" at the point $x = \sqrt{2}$. Moreover $f(1) = f(2) = 3/2$. From the elementary calculus it follows that $\sqrt{2} \leq f(x) \leq 3/2$ for $1 \leq x \leq 2$. By induction, for all $n \in \mathbb{N}$, we have $a_{n+1} = f(a_n) \in [\sqrt{2}, 3/2]$ – a small subinterval of $[1, 2]$.

b) We have

$$\varepsilon_{n+1} = a_{n+1} - \sqrt{2} = \frac{1}{2} \left(\varepsilon_n + \sqrt{2} + \frac{2}{\varepsilon_n + \sqrt{2}} \right) - \sqrt{2} = \frac{\varepsilon_n^2}{2a_n}.$$

Since $a_n \geq 1$ and $|\varepsilon_n| \leq 1$ (a very rough estimate), we get $|\varepsilon_{n+1}| \leq |\varepsilon_n|/2$ for all n . By induction, $|\varepsilon_n| \leq 2^{1-n}$, hence $\varepsilon_n \rightarrow 0$.