MATH 3283W. Sequences, Series, and Foundations: Writing Intensive. Spring 2009

Homework 2. Problems and Solutions

I. Writing Intensive Part

1 (5 points). Let $f(x)$ be a continuous function on the segment $[0,1]$, such that $f(0) < 0 < f(1)$. Show that $f(c) = 0$ for some $c \in (0,1)$.

Solution. Denote $S = \{x \in [0,1] : \ f(x) < 0\}$. This set is nonempty, because it contains 0, and bounded. Set $c = \sup S \in [0,1]$. Consider possible cases.

(i) $f(c) < 0$. Since f is continuous on $[0,1]$, there is a small $\varepsilon > 0$ such that $f(x) < 0$ on the set $S_1 = (c-\varepsilon, c+\varepsilon) \cap [0,1]$. This means $S_1 \subseteq S$. The inequality $f(1) > 0$ implies that S_1 does not contain 1, hence $c + \varepsilon \le 1$. Now $c < c + \varepsilon = \sup S_1 \le \sup S = c$. This contradiction shows that we cannot have $f(c) < 0$.

(ii) $f(c) > 0$. In this case, there is a small $\varepsilon > 0$ such that $f(x) > 0$ on the set $S_1 = (c-\varepsilon, c+\varepsilon) \cap [0,1]$. Since $f(0) < 0$, we must have $c - \varepsilon \ge 0$. By Definition 1.5 on p. 30, c is an upper bound of S. But S and S_1 do not intersect, which means S has no point in $(c-\varepsilon, c]$, which in turn implies that $c - \varepsilon$ is also an upper bound of S, which contradicts to the definition of $c = \sup S$ as the *least upper bound.*

(iii) $f(c) = 0$ - this is the only possibility left.

Another way. Partially following Exercise 6.20 on p.73, we can define a sequence of closed intervals $[0,1] = C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots$, where $C_{n+1} = [a_{n+1}, b_{n+1}]$ is the half of the interval $C_n = [a_n, b_n]$, such that $f(a_{n+1}) \le 0 \le f(b_{n+1})$. Then the sequence $\{a_n\}$ is non-decreasing, $\{b_n\}$ is non-increasing, and they have same limit c, because $b_n = a_n + 2^{-n}$. From $f(a_n) \le 0 \le f(b_n)$ and continuity of f it follows $f(c) = \lim f(a_n) \le 0 \le \lim f(b_n) = f(c)$, i.e. $f(c) = 0$.

2 (7 points). By one of equivalent definitions, a set $K \subset \mathbb{R}^1$ is *closed* is for any convergent sequence $\{s_n\}$, from $s_n \in K$ for all n it follows $L = \lim s_n \in K$. Let $K_1 \supset K_2 \supset K_3 \supset \dots$ be a sequence of bounded, nonempty, closed subsets in $\mathbb{R}^1$. Show that the intersection of all K_j is nonempty, i.e. there is a point $x \in K_j$ for all j.

Solution. For each $n \in \mathbb{N}$, take a point $s_n \in K_n$. By the Bolzano-Weierstrass theorem, there is a subsequence $\{s_{n_i}\}$ which converges to a point

$x \in \mathbb{R}^1$. For each $j \in \mathbb{N}$, there is a number i_0 (depending on j) such that $n_i \geq j$ for all $i \leq i_0$. This means $s_{n_i} \in K_{n_i} \subseteq K_j$ for all $i \geq i_0$. Thus the "restricted" sequence $\{s_{n_i},\ i \geq i_0\}$ is contained in K_j, and since K_j is closed, $x = \lim s_{n_i} \in K_j$. This is true for each $j \in \mathbb{N}$, so that $x \in \cap K_j$.

3 (8 points). This is an extension of Exercise 2.18 on p.13. Consider a function

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where each of $a_0, \ldots, a_n$ is an integer (some of them may be non-positive), and $a_n \geq 1$.

(a) Show that there is $k_0 \in \mathbb{N}$ such that $p(k) \geq 2$ for each $k \geq k_0$.

(b) Show that there is an integer $k \geq k_0$ such that $p(k)$ is not a prime number. *Hint.* Consider a polynomial $q(y) = p(k_0 + y)$.

Solution. (a) We have

$$x^{-n} p(x) = a_0 x^{-n} + a_1 x^{1-n} + \cdots + a_{n-1} x^{-1} + a_n \to a_n \geq 1 \quad \text{as} \quad n \to \infty.$$

Therefore, there is $N > 0$ such that $x^{-n} p(x) \geq 1/2$ for all $x \geq N$. Choose an integer $k_0 \geq \max\{N, 4\}$. Then for all $k \geq k_0$, we have $p(k) \geq k^n/2 \geq k/2 \geq 2$.

(b) We can write

$$q(y) = p(k_0 + y) = b_0 + b_1 y + b_2 y^2 + \cdots + b_n y^n,$$

where $b_0 = q(0) = p(k_0) \geq 2$, and $b_n = a_n \geq 1$. Take $y = m b_0$. Then $q(y) = p(k_0 + m b_0) = b_0 p_1(m)$, where

$$p_1(m) = 1 + b_1 m + b_2 b_0 m^2 + \cdots + b_n b_0^{n-1} m^n$$

is a polynomial of degree n. As in (a), there is a large number $m_0 \in \mathbb{N}$ such that $p_1(m_0) \geq 2$. Then for $k = k_0 + m_0 b_0$, we have

$$p(k) = b_0 p_1(m_0), \quad \text{where} \quad b_0 \geq 2 \quad \text{and} \quad p_1(m_0) \geq 2,$$

so that $p(k)$ is not a prime number.

II. General Part

4 (4 points). Exercise 5.6 b) and i). Find

$$\lim_{n\to\infty} n^{(1/\ln n)}, \qquad \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n.$$

Solution. Since $n^{(1/\ln n)} = \left(e^{\ln n}\right)^{(1/\ln n)} = e^1 = e$ for all n, the first limit is e. The second one,

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} = e^{-1}.$$

5 (3 points). Using the formula $a-b=(a^2-b^2)/(a+b)$, find

$$\lim_{n\to\infty}(\sqrt{n^4+n^3}+\sqrt{n^4-n^3}-2n^2).$$

Solution. Denote $\alpha_n = 1/n$. Then we can rewrite the above limit in the form

$$\begin{aligned}
&\lim_{n\to\infty}\frac{(1+\alpha_n)^{1/2}+(1-\alpha_n)^{1/2}-2}{\alpha_n^2}\\
=\ &\lim_{n\to\infty}\frac{1}{(1+\alpha_n)^{1/2}+(1-\alpha_n)^{1/2}+2}\cdot\frac{\left[(1+\alpha_n)^{1/2}+(1-\alpha_n)^{1/2}\right]^2-2^2}{\alpha_n^2}\\
=\ &\lim_{n\to\infty}\frac{(1-\alpha_n^2)^{1/2}-1}{2\alpha_n^2} = \lim_{n\to\infty}\frac{(1-\alpha_n^2)-1}{2\alpha_n^2\left[(1-\alpha_n^2)^{1/2}+1\right]} = -\frac{1}{4}.
\end{aligned}$$

Remark. There is a shorter way based on the formula

$$\left[f(x+h)+f(x-h)-2f(x)\right]/h^2 \to f''(x) \text{ as } h\to 0^+.$$

In our case, $f(x)=x^{1/2}$, $h=\alpha_n$, and $x=1$.

6 (3 points). Let $\{s_n\}$ be a convergent sequence with $\lim s_n = L$. Show that

$$\lim_{n\to\infty}\frac{s_1+s_2+\cdots+s_n}{n} = L.$$

Solution. Replacing s_n by $s_n - L$, we can assume that $\lim s_n = L = 0$. By definition, for arbitrary $\varepsilon > 0$, there is $m_0 \in \mathbb{N}$ such that $|s_k| < \varepsilon/2$ for all $k \geq m_0$. Then choose $n_0 \geq m_0$ large enough, such that $n_0 > |s_1+s_2+\cdots+s_{m_0-1}|\cdot 2/\varepsilon$. For arbitrary $n \geq n_0$, we have

$$\frac{|s_1+s_2+\cdots+s_{m_0-1}|}{n} \leq \frac{|s_1+s_2+\cdots+s_{m_0-1}|}{n_0} < \frac{\varepsilon}{2}.$$

Moreover, $|s_k| < \varepsilon/2$ for $k = m_0, m_0 + 1, \ldots, n$. Therefore,

$$\left|\frac{s_1 + s_2 + \cdots + s_n}{n}\right| \le \frac{|s_1 + s_2 + \cdots + s_{m_0-1}|}{n} + \frac{|s_{m_0}| + \cdots + |s_n|}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the desired property.

7 (5 points). Let $a_1 = 1$, and $a_{n+1} = (1 + a_n)^{-1}$ for $n = 1, 2, 3, \ldots$. Find $\lim a_n$. *Hint.* This sequence is NOT monotonic. Consider the differences between a_n and roots of the equation $x = (1 + x)^{-1}$.

Solution. The equation $x = (1 + x)^{-1}$, i.e. $x^2 + x - 1 = 0$ has a positive root $x = (\sqrt{5} - 1)/2$. We claim that this is the limit of a_n. Indeed, for $k = 1, 2, \ldots$

$$|a_{k+1} - x| = \left|\frac{1}{1 + a_k} - \frac{1}{1 + x}\right| = \frac{|a_k - x|}{(1 + a_k)(1 + x)} \le \frac{|a_k - x|}{1 + x}.$$

By induction,

$$|a_n - x| \le \frac{|a_1 - x|}{(1 + x)^{n-1}} \to 0 \quad \text{as} \quad n \to \infty.$$

Remark. There is a general formula for Fibonacci numbers: $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n = 1, 2, \ldots$, which is

$$F_n = \frac{\varphi^n + (1 - \varphi)^n}{\sqrt{5}}, \quad \text{where} \quad \varphi = \frac{1 + \sqrt{5}}{2}.$$

This was proved in class. Comparing the recursive relations for a_n and F_n, one can see that $a_n = F_n/F_{n+1} \to 1/\varphi = x$ as $n \to \infty$.