MATH 3283W. Sequences, Series, and Foundations: Writing Intensive. Spring 2009

Homework 3. Problems and Solutions

I. Writing Intensive Part

1. Let $f(x)$ be a continuous function on the segment $[0,1]$. Show that f is *uniformly continuous* on $[0,1]$, which means: $\forall\varepsilon > 0$, $\exists\delta > 0$, such that from $x, y \in [0,1]$ and $|x-y| < \delta$ it follows $|f(x) - f(y)| < \varepsilon$.

Solution. Suppose this statement is false. Then $\exists\varepsilon > 0$ such that $\forall\delta > 0$ there are $x, y \in [0,1]$ with $|x-y| < \delta$ such that $|f(x) - f(y)| \geq \varepsilon$. Choose a sequence $0 < \delta_j \to 0$ as $j \to \infty$, and the corresponding $x_j, y_j \in [0,1]$ with $|x_j - y_j| < \delta_j$ such that $|f(x_j) - f(y_j)| \geq \varepsilon$. By the Bolzano–Weierstrass theorem, the sequence $\{x_j\}$ contains a convergent subsequence: $x_{j_k} \to x_0 \in [0,1]$ as $k \to \infty$. Then also $y_{j_k} \to x_0$ as $k \to \infty$. Since $f(x)$ is continuous at x_0, we then have $f(x_{j_k}) - f(y_{j_k}) \to f(x_0) - f(x_0) = 0$, so that $|f(x_{j_k}) - f(y_{j_k})| < \varepsilon$ for large enough k. This contradiction shows that the given statement is true.

2. From Problem 6 in Homework 1 it follows

$$a_n = \left(1 + \frac{1}{n}\right)^n < e < b_n = \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Determine for which $n \in \mathbb{N}$ we have

$$c_n = \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} > e.$$

Solution. We claim that $c_n > e$ for all $n \in \mathbb{N}$, or equivalently,

$$\ln c_n = \left(n + \frac{1}{2}\right) \ln\left(1 + \frac{1}{n}\right) > \ln e = 1.$$

One can prove this inequality using Taylor's formula for $\ln(1+x)$. In order to simplify calculations, we write

$$\ln c_n = \frac{2n+1}{2} \ln\left(\frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}\right) = \frac{\ln(1+x_n) - \ln(1-x_n)}{2x_n},$$

where $x_n = \frac{1}{2n+1} > 0$. By Cauchy's Mean Value Theorem, applied with $f(x) = \ln(1+x) - \ln(1-x)$ and $g(x) = 2x$, for each n there is $y_n \in (0, x_n)$ such that

$$\ln c_n = \frac{f(x_n) - f(0)}{g(x_n) - g(0)} = \frac{f'(y_n)}{g'(y_n)} = \frac{(1+y_n)^{-1} - (1-y_n)^{-1}}{2} = \frac{1}{1-y_n^2} > 0.$$

3. Verify whether or not the sequence

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

is convergent.

Solution. By the mean value theorem

$$\ln(n+1) - \ln n = \frac{1}{x_n}, \quad \text{where} \quad n < x_n < n+1.$$

Therefore,

$$0 < s_n - s_{n+1} = \ln(n+1) - \ln n - \frac{1}{n+1} = \frac{1}{x_n} - \frac{1}{n+1} < \frac{1}{n} - \frac{1}{n+1}.$$

For arbitrary $n, k \in \mathbb{N}$,

$$\begin{aligned} 0 &< s_n - s_{n+k} = (s_n - s_{n+1}) + (s_{n+1} - s_{n+2}) + \cdots + (s_{n+k-1} - s_{n+k}) \\ &< \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{n+k-1} - \frac{1}{n+k}\right) \\ &= \frac{1}{n} - \frac{1}{n+k} < \frac{1}{n}. \end{aligned}$$

Hence $\{s_n\}$ is a Cauchy sequence, and it is convergent.

II. General Part

4. For $\lambda = const > 0$, evaluate the improper integrals

$$A(\lambda) = \int_0^\infty e^{-\lambda x} \cos x \, dx, \quad B(\lambda) = \int_0^\infty e^{-\lambda x} \sin x \, dx.$$

Solution. Using Euler's formula $e^{ix} = \cos x + i \sin x$, we obtain

$$\begin{aligned}
A(\lambda) + iB(\lambda) &= \int_0^\infty e^{-\lambda x}(\cos x + i \sin x) dx = \int_0^\infty e^{-\lambda x} e^{ix} dx \\
&= \int_0^\infty e^{-(\lambda - i)x} dx = \frac{-1}{\lambda - i} \cdot e^{-(\lambda-i)x}\Big|_{x=0}^{x=+\infty} \\
&= \frac{1}{\lambda - i} = \frac{\lambda + i}{(\lambda - i)(\lambda + i)} = \frac{\lambda + i}{\lambda^2 + 1}.
\end{aligned}$$

Comparing the real and imaginary parts, we find

$$A(\lambda) = \int_0^\infty e^{-\lambda x} \cos x \; dx = \frac{\lambda}{\lambda^2 + 1}, \quad B(\lambda) = \int_0^\infty e^{-\lambda x} \sin x \; dx = \frac{1}{\lambda^2 + 1}.$$

5. Check whether or not the series

$$\sum_{n=0}^{\infty} \ln\left(1 + 2^{-2^n}\right)$$

converges. If yes, find its sum.

Solution. The convergence follows easily from the comparison $\ln\left(1 + 2^{-2^n}\right) \leq 2^{-2^n}$. In order to find the explicit expression for the sum S, note that $S = \lim S_N$, where $S_N = \sum_{n=0}^{N} \ln\left(1 + 2^{-2^n}\right) = \ln T_N$, and

$$T_N = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{2^4}\right) \cdot \dots \cdot \left(1 + \frac{1}{2^{2^N}}\right).$$

It is easy to see that

$$\begin{aligned}
\left(1 - \frac{1}{2}\right) T_N &= \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{2^4}\right) \cdot \dots \cdot \left(1 + \frac{1}{2^{2^N}}\right) \\
&= \left(1 - \frac{1}{2^2}\right)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{2^4}\right) \cdot \dots \cdot \left(1 + \frac{1}{2^{2^N}}\right) \\
&= \left(1 - \frac{1}{2^4}\right)\left(1 + \frac{1}{2^4}\right) \cdot \dots \cdot \left(1 + \frac{1}{2^{2^N}}\right) = \cdots \\
&= \left(1 - \frac{1}{2^{2^N}}\right)\left(1 + \frac{1}{2^{2^N}}\right) = \left(1 - \frac{1}{2^{2^{N+1}}}\right) \to 1 \text{ as } N \to \infty.
\end{aligned}$$

Hence $T_N \to 2$, and $S = \lim \ln T_N = \ln 2$.

6. Let $A_1 \geq A_2 \geq \cdots \geq A_n \geq \cdots \geq 0$. Show that the series $\sum\limits_{n=1}^{\infty} A_n$ converges if and only if the series $\sum\limits_{n=1}^{\infty} B_n$ converges, where $B_n = 2^n A_{2^n}$, i.e. $B_1 = 2A_2,\ B_2 = 4A_4$, etc.

Solution. By definition, the series $\sum A_n$ converges if there exists the limit of partial sums:

$$\exists S = \lim_{N\to\infty} S_N, \quad S_N = \sum_{n=1}^{N} A_n.$$

Since $A_n \geq 0$ for all n, the sequence $\{S_N\}$ is non-decreasing. If the sequence S_N converges to the limit $S < \infty$, then any subsequence, in particular $T_n = S_{2^n}$, converges to the same limit. Using monotonicity, it is easy to show that from the convergence of T_n it follows the convergence fo S_N (to the same limit).

Further, one can treat T_n as the n-th partial sum for the series $\sum C_n$, where $T_1 = S_2$, and

$$C_n = T_n - T_{n-1} = A_{2^{n-1}+1} + A_{2^{n-1}+2} + \cdots + A_{2^n} \quad \text{for} \quad n = 2, 3, \ldots.$$

Here the right side contains 2^{n-1}, which are estimated by monotonicity as follows:

$$2^{n-1}A_{2^{n-1}} \leq C_n \leq 2^{n-1}A_{2^n}, \quad \text{i.e.} \quad B_{n-1} \leq C_n \leq \frac{1}{2}B_n.$$

By the comparison test, $\sum B_n$ converges $\Longleftrightarrow$ $\sum C_n$ converges, i.e. the sequence of partial sums $\{T_n\}$ converges $\Longleftrightarrow$ $\{S_N\}$ converges $\Longleftrightarrow$ $\sum A_n$ converges.