

MATH 3283W. Sequences, Series, and Foundations:  
Writing Intensive. Spring 2009

**Homework 4. Problems and Solutions**

1. Show that the number  $e$  is irrational.

**Solution.** If we assume that  $e$  is rational, i.e.  $r = \frac{m}{n}$  for some  $m, n \in \mathbb{N}$ , then  $n! \cdot e \in \mathbb{N}$ . By Taylor's expansion of  $e^x$  at the point  $x = 1$ ,

$$e = S_n + T_n, \quad \text{where} \quad S_n = \sum_{k=0}^n \frac{1}{k!}, \quad T_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}.$$

Obviously  $n! \cdot S_n \in \mathbb{N}$ . Together with  $n! \cdot e \in \mathbb{N}$ , this implies  $n! \cdot T_n = n! \cdot e - n! \cdot S_n \in \mathbb{N}$ . However,

$$\begin{aligned} 0 < n! \cdot T_n &= n! \cdot \sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots \\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots = \frac{1}{n+1} \cdot \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n} \leq 1. \end{aligned}$$

Since  $n! \cdot T_n$  lies strictly between 0 and 1, it cannot be integer. This contradiction shows that  $e$  is irrational.

2. Let numbers  $r \in \mathbb{N}$  and  $p \in (0, 1)$  be given. Define

$$p_k = \binom{k-1}{r-1} p^r q^{k-r} \quad \text{for } k = r, r+1, r+2, \dots, \quad \text{where } q = 1 - p.$$

Show that

$$\sum_{k=r}^{\infty} p_k = 1, \quad \text{and evaluate } \mu_1 = \sum_{k=r}^{\infty} k p_k \quad \text{and} \quad \mu_2 = \sum_{k=r}^{\infty} k^2 p_k.$$

**Solution.** We have

$$\begin{aligned} \binom{k-1}{r-1} &= \binom{k-1}{k-r} = \frac{(k-1)(k-1) \cdot \dots \cdot (r+1) \cdot r}{k-r} \\ &= (-1)^{k-r} \frac{(-r)(-r-1) \cdot \dots \cdot (1-k)}{k-r} = (-1)^{k-r} \binom{-r}{k-r}. \end{aligned}$$

Therefore, we can write

$$p_k = \binom{-r}{k-r} p^r (-q)^{k-r}, \quad \text{where} \quad \binom{a}{j} = \frac{a(a-1)(a-2) \cdot \dots \cdot (a-j+1)}{j!}.$$

Note that this notation makes sense for any **real**  $a$ . Using the formula

$$(1+x)^a = \sum_{j=0}^{\infty} \binom{a}{j} x^j \quad \text{for } |x| < 1,$$

we can calculate the generating function (substitute  $k = r + j$ )

$$\phi(t) = \sum_{k=r}^{\infty} p_k t^k = (pt)^r \sum_{j=0}^{\infty} \binom{-r}{j} (-qt)^j = (pt)^r (1 - qt)^{-r}.$$

We see that

$$\sum_{k=r}^{\infty} p_k = \phi(1) = p^r (1 - q)^{-r} = p^r p^{-r} = 1.$$

Differentiating the equality  $\ln \phi(t) = r \ln(pt) - r \ln(1 - qt)$ , we get

$$\frac{\phi'}{\phi} = \frac{r}{t} + \frac{rq}{1 - qt}, \quad \frac{\phi''}{\phi} - \left(\frac{\phi'}{\phi}\right)^2 = -\frac{r}{t^2} + \frac{rq^2}{(1 - qt)^2}.$$

The series  $\sum p_k t^k$  converges for  $|t| < 1/q$ , so that  $t = 1$  lies inside the interval of convergence, and we can differentiate this series at the point  $t = 1$ . This implies

$$\begin{aligned} \mu_1 &= \sum_{k=r}^{\infty} k p_k = \phi'(1) = r + \frac{rq}{1 - q} = \frac{r}{p}, \\ \mu_2 &= \sum_{k=r}^{\infty} k^2 p_k = \phi''(1) + \phi'(1) = -r + \frac{rq^2}{(1 - q)^2} + \frac{r^2}{p^2} + \frac{r}{p} \\ &= \frac{r(-p^2 + q^2 + r + p)}{p^2} = \frac{r(r + 1 - p)}{p^2}. \end{aligned}$$

3. Define

$$p_k(x) = e^{-x} \cdot \frac{x^k}{k!} \quad \text{for} \quad k = 0, 1, 2, \dots \quad \text{and} \quad x \in \mathbb{R}.$$

Find an analytic expression for the function

$$f(x) = \sum_{k=0}^{\infty} \frac{p_k(x)}{k + 1}.$$

Moreover, write  $f(x)$  as a power series in  $x$  and determine its radius of convergence.

**Solution.** Substituting  $j = k + 1$ , we get

$$\begin{aligned} f(x) &= e^{-x} \cdot \sum_{k=0}^{\infty} \frac{x^k}{(k + 1)!} = \frac{e^{-x}}{x} \cdot \sum_{j=1}^{\infty} \frac{x^j}{j!} = \frac{e^{-x}}{x} \cdot (e^x - 1) = \frac{1 - e^{-x}}{x} \\ &= \frac{1}{x} \left[ 1 - \left( 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \right) \right] = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n + 1)!}. \end{aligned}$$