MATH 3283W. Sequences, Series, and Foundations: Writing Intensive. Spring 2009
Drill Problems to Midterm Exam #1, March 5.

The Exam will consist of 5 problems, some of which may be similar to those in the following list. The problems in Ch. 1–3 are also recommended.

#1. Let S be a nonempty set in $\mathbb{R}^1$, and let S_1 be the set of all limit points of S, i.e. each point in S_1 is a limit of a sequence $\{s_n\} \subseteq S$. Show that the set $\overline{S} = S \cup S_1$ (which is called the *closure* of S) is closed, i.e. it contains all its limit points.

#2. Let $f(x)$ be a continuous function on $[0,1]$, such that

$$0 < a \le f(x) \le b < 1 \qquad \text{for all} \quad x \in [0,1].$$

Here a and b are given constants. Show that there is a point $x \in (0,1)$ such that $f(x) = x$.

#3. Find the limits

$$(a)\ \lim_{n\to\infty} \sqrt[n]{n}, \quad (b)\ \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}, \quad (c)\ \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}.$$

Here (a) is a problem on p. 62 in the textbook. (c) is really hard: it uses the relations

$$\ln k \le \int_k^{k+1} \frac{dt}{t} \le \ln(k+1).$$

#4. Consider the sequence $s_1 = 0$, $s_2 = 1$, and $s_{n+2} = \frac{1}{2}(s_n + s_{n+1})$ for $n = 1, 2, \ldots$. Find $\lim s_n$.

#5. (Ex. 6.11 on p. 70). For which values of the real number c does the recursively defined sequence

$$a_0 = c, \quad a_{n+1} = \frac{a_n^2 + 2}{3} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

(**a**) converge to 1?
(**b**) converge to 2?
(**c**) diverge?

Hint. Consider $f(x) = (x^2 + 2)/3$ and describe the sets $\{f(x) > x\}$ and $\{f(x) < x\}$.