MATH 3283W. Sequences, Series, and Foundations: Writing Intensive. Midterm Exam #1. March 5, 2009. Problems and Solutions

#1. Rewrite the sentence $P\&Q = P \vee Q$ in a simpler form.

Solution. There are many ways to do it. For example, one can just compare the truth tables. The answer is $P = Q$.

#2. Let $f(x)$ be a continuous function on $[0,1]$, such that

$$0 < a \le f(x) \le b < 1 \qquad \text{for all} \quad x \in [0,1].$$

Here a and b are given constants. Show that there is a point $x \in (0,1)$ such that $f(x) = x$.

Solution. We can apply the result of Problem 1 in Homework 2 to the function $F(x) = x - f(x)$. This is a continuous function on $[0,1]$, such that

$$F(0) = -f(0) \le -a < 0, \ \text{ and } \ F(1) = 1 - f(1) \ge 1 - b > 0.$$

Therefore, there is $c \in (0,1)$ such that $F(c) = 0$, i.e. $f(c) = c$.

#3. Using the *binomial formula*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \ \text{ where } \ \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

$(0! = 1! = 1,\ k! = 1 \cdot 2 \cdot \cdots \cdot k$ for $k \ge 2)$, simplify the expression

$$S_n = \binom{n}{1} + 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} + \cdots + n \cdot \binom{n}{n}.$$

Solution. The shortest way is to notice that S_n is the derivative of $(1+x)^n$ at the point $x = 1$, so that $S_n = n \cdot 2^{n-1}$. Alternatively, one can write

$$k \cdot \binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = n \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = n \cdot \binom{n-1}{k-1}.$$

Therefore, by the binomial formula, with $n-1$ in place of n, we get

$$S_n = n \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} = n \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} = n \cdot 2^{n-1}.$$

#4. Find the limits

$$(a) \ \lim_{n\to\infty} \left(\sqrt[4]{n^4 + n^3} - n \right), \quad (b) \quad \lim_{n\to\infty} \frac{2^n n!}{n^n}.$$

Solution. (a) One can use L'Hôpotal's Rule for the form 0/0 with $x = 1/n \to 0$:

$$\lim_{n\to\infty}\left(\sqrt[4]{n^4+n^3}-n\right) = \lim_{n\to\infty}\frac{(1+1/n)^{1/4}-1}{1/n} = \lim_{x\to 0}\frac{(1+x)^{1/4}-1}{x} = \frac{1}{4}.$$

Alternatively, one can use the formula $(a-b) = (a^4-b^4)/(a^3+a^2b+ab^2+b^3)$ with $a = 1+1/n,\ b = 1$.

(b). Denote $a_n = 2^n n!/n^n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n n!} = \frac{2}{\left(1+\frac{1}{n}\right)^n} \to \frac{2}{e} < 1 \text{ as } n\to\infty.$$

Therefore, for a fixed constant $c \in (2/e, 1)$, there is $n_0 \ge 1$ such that $a_{n+1}/a_n \le c$ for all $n \ge n_0$. By interation,

$$a_{n_0+k} \le a_{n_0}c^k \to 0 \text{ as } k\to\infty, \text{ so that } \lim a_n = 0.$$

One can also use the limit in Drill Problem #3(c), which is e (if you solved this problem). Then

$$\frac{2\cdot\sqrt[n]{n!}}{n} \to \frac{2}{e} < 1 \text{ and } a_n = \left(\frac{2\cdot\sqrt[n]{n!}}{n}\right)^n \to 0 \text{ as } n\to\infty.$$

#5. Consider the sequence

$$s_1 = 0,\ s_2 = 1, \text{ and } s_{n+2} = \frac{1}{3}(s_n + 2s_{n+1}) \text{ for } n = 1, 2, \ldots.$$

Determine whether this sequence is convergent or divergent. If convergent, find its limit.

Solution. Following the general method (given in class) compose the quadratic equation $r^2 = \frac{1}{3}(1+2r)$, which has roots $r_1 = 1$ and $r_2 = -1/3$. Then $s_n = c_1 r_1^n + c_2 r_2^n$ satisfy the given recursive equation for any constants c_1 and c_2. From the initial conditions $s_1 = 0$ and $s_2 = 1$, we find $c_1 = 3/4$ and $c_2 = 9/9$. The answer is

$$s_n = \frac{9}{4}\left(-\frac{1}{3}\right)^n + \frac{3}{4}.$$

It is also possible to solve this problem without using the general method. The sequence $t_n = s_{n+1} - s_n$ satisfies

$$t_1 = 1, \text{ and } t_{n+1} = -\frac{1}{3}t_n \text{ for } n = 1, 2, \ldots.$$

By iteration, $t_n = (-1/3)^{n-1}$. Then for $n \ge 2$,

$$\begin{aligned} s_n &= (s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_2 - s_1) + s_1 = t_{n-1} + t_{n-2} + \cdots + t_1 \\ &= (-1/3)^{n-2} + (-1/3)^{n-3} + \cdots + 1 = \frac{(-1/3)^{n-1} - 1}{(-1/3) - 1} = \frac{9}{4}\left(-\frac{1}{3}\right)^n + \frac{3}{4}. \end{aligned}$$