MATH 3283W. Sequences, Series, and Foundations:
Writing Intensive.
Midterm Exam #2. April 16, 2009. Problems and Solutions

Determine whether or not each of the following three series is convergent or divergent:

#1.

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{\ln n}}.$$

#2.

$$\sum_{n=1}^{\infty} \left(\frac{1+\cos n}{2+\cos n}\right)^{2n}.$$

#3.

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(\ln n))^{\ln n}}.$$

Solution #1. By L'Hôpital's Rule,

$$\lim_{n\to\infty} \ln\left(\sqrt[n]{\ln n}\right) = \lim_{n\to\infty} \frac{\ln(\ln n)}{n} = \lim_{x\to+\infty} \frac{\ln(\ln x)}{x} = \lim_{x\to+\infty} \frac{1}{x\ln x} = 0.$$

Therefore, $\sqrt[n]{\ln n} \to 1$ as $n \to \infty$, and for the given series $\sum a_n$ we have $\lim a_n = 1 \neq 0$, which in turn implies that the series is **divergent.**

Solution #2. We have

$$a_n = \left(\frac{1+\cos n}{2+\cos n}\right)^{2n} = \left(1 - \frac{1}{2+\cos n}\right)^{2n} \leq \left(1-\frac{1}{3}\right)^{2n} = \left(\frac{4}{9}\right)^{n} = b_n.$$

Since the geometric series $\sum b_n$ converges, the given series $\sum a_n$ **converges** by Comparison Test.

Solution #3. We can write

$$a_n = \frac{1}{(\ln(\ln n))^{\ln n}} = \frac{1}{e^{\ln n\cdot\ln(\ln(\ln n))}} = \frac{1}{n^{\ln(\ln(\ln n))}} \leq \frac{1}{n^2} = b_n$$

for large enough n, namely for $n > \exp(\exp(\exp 2)))$. We know that the series $\sum b_n$ converges (this is the case of $\sum \frac{1}{n^p}$ with $p > 1$). Hence the given series $\sum a_n$ **converges** by Comparison Test.

#4. Show that the following series convergent, and find its sum:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

Solution. The N-th partial sum

$$\begin{aligned} S_N &= \sum_{n=2}^{N} \frac{1}{n^2-1} = \frac{1}{2}\sum_{n=2}^{N}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\ &= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{N-1}-\frac{1}{N+1}\right)\right] \\ &= \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{N}-\frac{1}{N+1}\right) \to \frac{3}{4} \qquad \text{as} \quad N \to \infty. \end{aligned}$$

This means that the sum of this series is $3/4$.

#5. Find the coefficients a_n in the power series

$$\frac{1}{1+x-2x^2} = \sum_{n=0}^{\infty} a_n x^n,$$

and determine its radius of convergence.

Solution. We decompose the given expression into partial fractions:

$$\frac{1}{1+x-2x^2} = \frac{1}{(1-x)(1+2x)} = \frac{1/3}{1-x} + \frac{2/3}{1+2x} = \frac{1/3}{1-x} + \frac{1/3}{1/2+x}.$$

Then use the power series expansion

$$\frac{1}{a-x} = \sum_{n=0}^{\infty} \frac{x^n}{a^{n+1}} \qquad \text{for} \quad |x| < |a|.$$

This yields

$$\frac{1}{1+x-2x^2} = \frac{1}{3}\sum_{n=0}^{\infty} x^n + \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-x)^n}{(1/2)^{n+1}} = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = \frac{1}{3}\left[1 + (-1)^n 2^{n+1}\right]$.