MATH 4512. Differential Equations with Applications.
Final Exam. May 13, 2009, VinH 1

80 points are distributed between 8 problems, 10 points each.
You have 120 minutes (1:30 pm – 3:30 pm) to work on these problems.
No books, no notes, and no calculators.

1. Find a solution of the equation

$$x \cdot \frac{dy}{dx} + y = 1, \qquad x > 0,$$

such that $y(1) = 2$.

2. Find the general solution of the equation

$$\frac{dy}{dx} - \frac{y}{x} = x \cos x.$$

3. Solve the initial value problem

$$y'' + y = 4t \sin t, \qquad y(0) = y'(0) = 0.$$

4. Find the general solution of the equation

$$y'' + 2y' + y = e^{-x} \ln x, \qquad x > 0.$$

5. Solve the initial value problem

$$y^{(4)} - 2y'' + y = 0, \qquad y(0) = y'(0) = y''(0) = 0, \quad y'''(0) = 1.$$

6. Find a series solution in powers of x of the equation

$$y'' - 2xy' + 8y = 0,$$

which satisfies the initial conditions $y(0) = 3,\ y'(0) = 0$.

7. Use Laplace transforms to solve the equation

$$y'' + y = \sin t + (\sin t) * y(t), \qquad \text{where} \quad (\sin t) * y(t) = \int_0^t \sin(t - \tau)\ y(\tau)\ d\tau,$$

with the initial conditions $y(0) = 0,\ y'(0) = 1$.

Hint.

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}, \qquad \mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \qquad \mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0).$$

8. Let $y(t)$ be a smooth function on $[0, \infty)$, such that

$$y'' + p(t)\ y' + q(t)\ y = 0 \qquad \text{for all} \quad t > 0, \qquad y(0) = 0, \quad y'(0) = 1,$$

where $p(t)$ and $q(t)$ are bounded continuous functions on $[0, \infty)$, and $q(t) < 0$ for all $t > 0$.
Show that $y(t) > 0$ for all $t > 0$.