MATH 4512. Differential Equations with Applications.
Final Exam. Problems and Solutions

1. Find a solution of the equation

$$x \cdot \frac{dy}{dx} + y = 1, \qquad x > 0,$$

such that $y(1) = 2$.

Solution. This is a linear equation, which can be written in the form $(xy)' = 1$. By integration, we get $xy = x + C$, and $y = 1 + Cx^{-1}$. From the initial condition $y(1) = 2$ it follows $C = 1$, hence $y = 1 + x^{-1}$.

2. Find the general solution of the equation

$$\frac{dy}{dx} - \frac{y}{x} = x \cos x.$$

Solution. This is also a linear equation with the integrating factor $\mu = x^{-1}$. We can write it in the form $(x^{-1}y)' = \cos x$. By integration, we get $x^{-1}y = \sin x + C$, and $y = (\sin x + C)x$.

3. Solve the initial value problem

$$y'' + y = 4t \sin t, \qquad y(0) = y'(0) = 0.$$

Solution. The homogeneous equation $Ly = y'' + y = 0$ has independent solutions $y_1 = \cos t$ and $y_2 = \sin t$. We can find a solution y_p to the given non-homogeneous equation in the form $y_p = \operatorname{Im} z_p$, where $z_p = t(At + B)e^{it}$ satisfies $Lz_p = 4te^{it}$. We have

$$\begin{aligned} Lz_p &= (D^2 + 1)\left[t(At + B)e^{it}\right] = e^{it}\left[(D + i)^2 + 1\right](At^2 + Bt) \\ &= e^{it}(D^2 + 2iD)(At^2 + Bt) = e^{it}(2A + 4Ait + 2Bi) = e^{it} \cdot 4t, \end{aligned}$$

hence $A = -i, \ B = 1$, and

$$y_p = \operatorname{Im} z_p = \operatorname{Im}\left[\left(-it^2 + t\right)(\cos t + i \sin t)\right] = t \sin t - t^2 \cos t.$$

Since $y = y_p$ satisfies the conditions $y(0) = y'(0) = 0$, the answer is $y = t \sin t - t^2 \cos t$.

4. Find the general solution of the equation

$$y'' + 2y' + y = e^{-x} \ln x, \qquad x > 0.$$

Solution. The homogeneous equation $Ly = y'' + 2y' + y = 0$ has independent solutions $y_1 = e^{-x}$ and $y_2 = xe^{-x}$. A particular solution to the equation $Ly = e^{-x} \ln x$ has the form $y_p = u_1 y_1 + u_2 y_2$, where

$$u_1' y_1 + u_2' y_2 = 0, \qquad u_1' y_1' + u_2' y_2' = e^{-x} \ln x,$$

i.e.

$$u_1' e^{-x} + u_2' x e^{-x} = 0, \qquad -u_1' e^{-x} + u_2'(1 - x)e^{-x} = e^{-x} \ln x.$$

The left side of the second equation is $u_2' e^{-x}$. Therefore, we obtain $u_2' = \ln x, \ u_1' = -xu_2' = -x \ln x$. By integration (we drop integration constants), $u_2 = x \ln x - x$,

$$u_1 = -\int x \ln x \, dx = -\frac{1}{2} \int \ln x \, d(x^2) = -\frac{x^2 \ln x}{2} + \frac{x^2}{4},$$

and

$$y_p = u_1y_1 + u_2y_2 = \left(-\frac{x^2\ln x}{2} + \frac{x^2}{4}\right)e^{-x} + (x\ln x - x)xe^{-x} = \left(\frac{\ln x}{2} - \frac{3}{4}\right)x^2e^{-x}.$$

The general solution to the given equation is $y = (C_1 + C_2x)e^{-x} + y_p$.

5. Solve the initial value problem

$$y^{(4)} - 2y'' + y = 0, \qquad y(0) = y'(0) = y''(0) = 0, \quad y'''(0) = 1.$$

Solution. Let $Y(s)$ denote the Laplace transform of $y(t)$. Then $(s^4 - 2s^2 + 1)Y - 1 = 0$, and

$$Y(s) = \frac{1}{s^4 - 2s^2 + 1} = \frac{1}{(s+1)^2(s-1)^2} = \frac{1}{4}\left[\frac{1}{(s+1)^2} + \frac{1}{(s-1)^2} + \frac{1}{s+1} - \frac{1}{s-1}\right],$$

which corresponds to the function

$$y(t) = \frac{1}{4}\left(te^{-t} + te^{t} + e^{-t} - e^{t}\right) = \frac{1}{2}(t\cosh t - \sinh t).$$

6. Find a series solution in powers of x of the equation

$$y'' - 2xy' + 8y = 0,$$

which satisfies the initial conditions $y(0) = 3,\ y'(0) = 0$.

Solution. One can deal with this problem in the same way as with problem #5 in Midterm 2. Up to a constant, y is a Hermite polynomial of degree 4 :

$$y(x) = \frac{1}{4}H_4(x) = 3 - 12x^2 + 4x^4.$$

7. Use Laplace transforms to solve the equation

$$y'' + y = \sin t + (\sin t) * y(t), \qquad \text{where} \quad (\sin t) * y(t) = \int_0^t \sin(t-\tau)\ y(\tau)\ d\tau,$$

with the initial conditions $y(0) = 0,\ y'(0) = 1$.

Solution. Denote $Y(s) = \mathcal{L}\{y\}$ – the Laplace transform of $y(t)$. Then

$$(s^2+1)Y(s) - 1 = \frac{1}{s^2+1} + \frac{1}{s^2+1}\cdot Y(s),$$

which implies

$$\left[(s^2+1)^2 - 1\right]Y(s) = s^2 + 2, \qquad (s^4 + 2s^2)Y(s) = s^2 + 2, \qquad Y(s) = s^{-2},$$

which corresponds to $y(t) = t$.

8. Let $y(t)$ be a smooth function on $[0,\infty)$, such that

$$y'' + p(t)\ y' + q(t)\ y = 0 \qquad \text{for all} \quad t > 0, \qquad y(0) = 0, \quad y'(0) = 1,$$

where $p(t)$ and $q(t)$ are bounded continuous functions on $[0,\infty)$, and $q(t) < 0$ for all $t > 0$. Show that $y(t) > 0$ for all $t > 0$.

Proof. From the initial conditions $y(0) = 0,\ y'(0) = 1$, it follows that $y(t) > 0$ for small $t > 0$. If we assume that $y(T) \le 0$ for some $T > 0$, then $y(t)$ attains its maximum on $[0,T]$ at some point $t_0 \in (0,T)$. We then have

$$y(t_0) > 0, \qquad y'(t_0) = 0, \qquad y''(t_0) \le 0, \qquad Ly(t_0) = y''(t_0) + p(t_0)\ y'(t_0) + q(t_0)\ y(t_0) < 0,$$

which contradicts to our assumption $Ly(t) = 0$ for all $t > 0$.