# MATH 4512. Differential Equations with Applications.
# Midterm Exam #1. March 4, 2009.
# Problems and Solutions

**1.** Solve the initial value problem

$$x\frac{dy}{dx} = 2y + \frac{3}{x}, \qquad y(1) = 0.$$

**Solution.**Rewrite the equation in the form $Ly = y' - 2x^{-1}y = 3x^{-2}$. This is a linear equation. The corresponding homogeneous equation $Ly = \frac{dy}{dx} - \frac{2y}{x}$ is separable, and it has a solution $y_1 = x^2$. One can find a particular solution of the initial equation in the form $Y = u \cdot y_1$. Then $LY = u'y_1 + uLy_1 = u'y_1 = u'x^2 = 3x^{-2}$, hence $u' = 3x^{-4}$, $u = -x^{-3}$, and $Y = u \cdot y_1 = -x^{-1}$. Therefore, the general solution is

$$y = Cy_1 + Y = Cx^2 - x^{-1}.$$

Since $y(1) = 0$, we have $C = 1$, and $y = x^2 - x^{-1}$.

**2.** Find the general solution of the equation

$$\frac{dy}{dx} - xy^2 = xy.$$

**Solution.** We have

$$\begin{aligned}\frac{dy}{dx} &= x(y^2 + y) \implies \frac{dy}{y(y+1)} = x\,dx \implies \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = x\,dx \\ &\implies \ln|y| - \ln|y+1| = \frac{1}{2}x^2 + C_0 \implies 1 + \frac{1}{y} = \frac{y+1}{y} = Ce^{-\frac{1}{2}x^2}.\end{aligned}$$

This argument does not work for $y = 0$ and $y = -1$, because the denominator $y(y+1) = 0$. We need to verify these cases separately. Both $y = 0$ and $y = -1$ are solutions, but $y = -1$ is already contained in the previous formula with $C = 0$.

**3.** Find the general solution of the equation

$$x\frac{dy}{dx} = y(\ln y - \ln x).$$

**Solution.** This is a homogeneous equation.

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x}\ln\frac{y}{x},\ u=\frac{y}{x} \implies u+x\frac{du}{dx}=u\,\ln u,\quad \frac{du}{u(\ln u-1)}=\frac{dx}{x},\\ \ln|\ln u-1| &= \ln|x|+C_0,\ \ \ln u-1=Cx,\ \ y=xu=x\,e^{Cx+1}.\end{aligned}$$

**4.** Solve the initial value problem

$$y''-7y'+6y=0,\qquad y(0)=2,\ y'(0)=7.$$

**Solution.** The characteristic equation $r^2-7r+6=0$ has two distinct roots $r_1=1$, $r_2=6$. Therefore, the general solution is $y(x)=c_1e^x+c_2e^{6x}$. The initial conditions give us

$$y(0)=c_1+c_2=2,\qquad y'(0)=c_1+6c_2=7.$$

Hence $c_1=c_2=1$, and $y(x)=e^x+e^{6x}$.

**5.** Find the general solution of the equation of the differential equation

$$Ly=y''+4y'+5y=e^{-2x}\sin x.$$

**Solution.** The characteristic equation $r^2+4r+5=(r+2)^2+1=0$ has roots $r_{1,2}=-2\pm i$. Therefore, $y_1=e^{-2x}\cos x$ and $y_2=e^{-2x}\sin x$ are independent solutions of $Ly=0$. Since the right side, which is exactly

$$y_2=e^{-2x}\sin x=e^{-2x}\cdot\frac{e^{ix}-e^{-ix}}{2i},$$

corresponds to the roots $r_{1,2}$ of multiplicity 1, we can find a particular solution of the given equation in the form

$$Y=xe^{-2x}u,\qquad \text{where}\quad u=u(x)=A\cos x+B\sin x.$$

We have

$$\begin{aligned}LY &= \left[(D+2)^2+1\right]\left(e^{-2x}xu\right)=e^{-2x}\left(D^2+1\right)(xu)\\ &= e^{-2x}\left[2u'+x(D^2+1)u\right]=e^{-2x}\cdot 2u'=e^{-2x}\sin x.\end{aligned}$$

From the equality $2u'=-2A\sin x+2B\cos x\equiv\sin x$ it follows $A=-\frac{1}{2}$, $B=0$. Finally, the general solution is

$$y=c_1y_1+c_2y_2+Y=e^{-2x}(c_1\cos x+c_2\sin x)-\frac{1}{2}x\cdot e^{-2x}\cos x.$$