MATH 4512. Differential Equations with Applications.
Midterm Exam #2. April 15, 2009.
Problems and Solutions

1. Consider a linear homogeneous equation $Ly = ay'' + by' + cy = 0$ with constant coefficients a, b, c, which are **strictly positive.** Show that for an arbitrary solution to this equation we have

$$y(t) \to 0 \qquad \text{as} \quad t \to +\infty.$$

Solution. We always have $y = c_1 y_1 + c_2 y_2$, where y_1 and y_2 are linearly independent solutions of the equation $Ly = 0$. Therefore, it suffices to show that $y_1(t),\ y_2(t) \to 0$ as $t \to +\infty$. There are three cases depending on the roots r_1 and r_2 of the *characteristic equation*

$$r_{1,2} = \frac{-b \pm \sqrt{D}}{2a}, \qquad \text{where} \quad D = b^2 - 4ac.$$

(i) $D > 0$. Since $\sqrt{D} < b$, both r_1 and r_2 are different and **strictly negative**. In this case, for $k = 1$ and 2, we have $y_k(t) = e^{r_k t} \to 0$ as $t \to +\infty$.

(ii) $D = 0$. In this case, $r_1 = r_2 = r = -b/2a < 0$, and both $y_1(t) = e^{rt}, y_2(t) = te^{rt} \to 0$ as $t \to +\infty$.

(iii) $D < 0$. In this case, $r_{1,2} = \lambda \pm i\mu$, where $\lambda = -b/2a < 0$. Then both $y_1(t) = e^{\lambda t}\cos(\mu t)$ and $y_2(t) = e^{\lambda t}\sin(\mu t)$ satisfy $|y_{1,2}(t)| \le e^{\lambda t} \to 0$ as $t \to +\infty$.

2. Find the general solution of the equation

$$Ly = t^2 y'' - ty' + y = 4t\ln t \qquad \text{for} \quad t > 0,$$

given that $y_1(t) = t$ and $y_2(t) = t\ln t$ are independent solutions to the homogeneous equation $Lu = 0$.

Solution. The general solution has the form $y = c_1 y_1 + c_2 y_2 + Y$, where a particular solution can be found by the method of of variation of parameters in the form $Y = u_1 y_1 + u_2 y_2$, where

$$u_1' y_1 + u_2' y_2 = 0, \qquad u_1' y_1' + u_2' y_2' = 4t^{-1}\ln t.$$

The right side here is $4t\ln t$ divided by t^2 – the coefficient of y''. After a minor simplification, we rewrite this system in the form

$$u_1' + u_2' \ln t = 0, \qquad u_1' + u_2'(1 + \ln t) = 4t^{-1}\ln t.$$

Subtracting both sides of the first equation from the second one, we get:

$$u_2' = 4t^{-1}\ln t, \qquad u_1' = -4t^{-1}(\ln t)^2.$$

Both u_1 and u_2 are obtained by integration with substitution $u = \ln t,\ du = t^{-1}dt$. We do not need to add the constants of integration, because eventually they will be added to the constants c_1 and c_2 in $y = c_1 y_1 + c_2 y_2 + Y$. Then

$$\begin{aligned} u_1 &= -4\int \frac{(\ln t)^2}{t}dt = -\frac{4}{3}(\ln t)^3, \quad u_2 = 4\int \frac{\ln t}{t}dt = 2(\ln t)^2, \\ Y &= u_1 y_1 + u_2 y_2 = -\frac{4t}{3}(\ln t)^3 + 2t(\ln t)^3 = \frac{2t}{3}(\ln t)^3. \end{aligned}$$

Finally, the general solution is

$$y = c_1 y_1 + c_2 y_2 + Y = (c_1 + c_2 \ln t)t + \frac{2t}{3}(\ln t)^3.$$

Remark. This is an Euler equation (see Problem 38 in Sec. 3.4, or Sec. 5.5). One can reduce it to an equation with constant coefficients by substitution $t = e^x$, so that $y(t) = y(e^x) = z(x) = z(\ln t)$. Then

$$\begin{aligned} y'(t) &= t^{-1}z'(\ln t), \qquad y''(t) = t^{-2}z''(\ln t) - t^{-2}z'(\ln t), \\ Ly &= z'' - 2z' + z = 4xe^x. \end{aligned}$$

The last equation has the form $(D-1)^2 z = 4xe^x$, where $D = d/dx$. The corresponding characteristic equation $(r-1)^2 = 0$ has a root $r = 1$ of multiplicity 2. One can find a particular solution in the form $Z = x^2(Ax + B)e^x$. From the equality

$$(D-1)^2 Z = (D-1)^2\left[(Ax^3 + Bx^2)e^x\right] = e^x(Ax^3 + Bx^2)'' = 4xe^x$$

we obtain $A = \frac{2}{3}$, $B = 0$. Therefore,

$$z = c_1 z_1 + c_2 z_2 + Z = (c_1 + c_2 x)e^x + \frac{2}{3}x^3 e^x.$$

Since $x = \ln t$, $e^x = t$, this expression has the same form as the above expression for $y = c_1 y_1 + c_2 y_2 + Y$.

3. Find the general solution of the equation

$$y^{(5)} + 8y''' + 16y' = 0.$$

Solution. The characteristic equation $r^5 + 8r^3 + 16r = r(r^2 + 4)^2 = 0$ has a simple root $r_1 = 0$ (of multiplicity 1), and two roots $r_{2,3} = \pm 2i$ of multiplicity 2. Therefore, the general solution is

$$y = c_1 + (c_2 + c_3 t)\cos(2t) + (c_4 + c_5 t)\sin(2t)$$

4. Find the coefficients a_n in the power series

$$\frac{x}{x^2 - 5x + 6} = \sum_{n=0}^{\infty} a_n x^n,$$

and determine its radius of convergence.

Solution. We can write

$$\begin{aligned} \frac{x}{x^2 - 5x + 6} &= \frac{2}{2-x} - \frac{3}{3-x} = \frac{1}{1 - x/2} - \frac{1}{1 - x/3} \\ &= \sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} a_n x^n, \end{aligned}$$

where $a_n = 2^{-n} - 3^{-n}$. This expansion holds true for $|x| < 2$, i.e. the radius of convergence is 2.

5. Find a series solution in powers of x of the equation

$$(1-x^2)y'' - xy' + 16y = 0,$$

which satisfies the initial conditions $y(0)=1,\ y'(0)=0$.

Solution. We can write and put into the equation the series

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Here the summation for y' and y'' actually starts from $n=1$ and $n=2$ correspondingly, but we prefer to have it from $n=0$, because the coefficients of x^{-1} and x^{-2} are 0. Then we do not need to consider the cases $n=0$ and $n=1$ separately. Collecting together the coefficients of x^n in the given equation, we get

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + 16a_n = 0, \qquad \text{i.e.} \quad a_{n+2} = \frac{n^2-16}{(n+2)(n+1)} a_n.$$

The initial conditions $y(0)=1,\ y'(0)=0$ imply $a_0=1,\ a_1=0$. From the recurrent relation for a_n it follows that $a_n=0$ for $n=1,3,5,7,\ldots$, and also for $n=6,8,10,\ldots$. The remaining two coefficients $a_2=-8$ and $a_4=8$, i.e. $y(x)=1-8x^2+8x^4$.

Remark. This is a Chebyshev polynomial of degree 4 :

$$y(x) = \cos(4t) = 1 - 2\sin^2(2t) = 1 - 8\sin^2 t\cos^2 t = 1 - 8\cos^2 t + 8\cos^4 t, \qquad \text{where} \quad \cos t = x.$$

Alternative way. Instead of comparing the coefficients in power series, on can use the following *Leibnitz formula* for the higher order derivative of a product:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} = f^{(n)} g + n f^{(n-1)} g' + \frac{n(n-1)}{2} f^{(n-2)} g'' + \cdots .$$

Denote $c_n = y^{(n)}(0) = a_n \cdot n!$. Applying the n-th order differentiation at the point $x=0$, we obtain

$$c_{n+2} - 2\binom{n}{2} c_n - \binom{n}{1} c_n + 16 c_n = 0, \qquad \text{i.e.} \quad c_{n+2} = (n^2-16) c_n.$$

Since $c_n = a_n \cdot n!$, we can rewrite this equality ias $a_{n+2}\cdot(n+2)! = (n^2-16)a_n \cdot n!$. After cancelation by $n!$, we arrive at the same expression as above.