Math 5652: Introduction to Stochastic Processes: Spring 2008

Homework Assignment 3 (due on Tuesday, March 11).

60 points are distributed between 5 problems, 12 points each. Problems 1–4 are taken from the textbook by R. Durrett (Exercises 9.12, 9.22, 9.38(a), and 9.44 in Section 1.9).

1. Consider the Markov chain with transition matrix:

$$\begin{pmatrix} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & 0 & 0 & 3/5 & 2/5 \\ \mathbf{2} & 0 & 0 & 1/5 & 4/5 \\ \mathbf{3} & 1/4 & 3/4 & 0 & 0 \\ \mathbf{4} & 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

(a) Compute p^2. **(b)** Find the stationary distributions of p and p^2. **(c)** Find the limit of $p^{2n}(x,x)$ as $n \to \infty$.

2. *Random walk on the circle.* Consider the points $1, 2, 3, 4$ to be marked on a ring. Let X_n be a Markov chain that moves to the right with probability p and to the left with probability $1-p$, subject to the rules that if X_n goes to the left from 1 it ends up at 4, and if X_n goes to the right from 4 it ends up at 1. Find **(a)** the transition probability for the chain, and **(b)** the limiting amount of time the chain spends at each site.

3. A criminal named X and a policeman named Y move between three possible hideouts according to independent Markov chains X_n and Y_n with transition probabilities:

$$p_X = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \qquad \text{and} \qquad p_Y = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}.$$

At time $T = \min\{n : X_n = Y_n\}$ the game is over and the criminal is caught. Suppose $X_0 = i$ and $Y_0 = j \neq i$. Find the expected value of T.

4. *Knight's random walk.* If we represent our chessboard as $S = \{(i,j) : 1 \le i, j \le 8\}$ then a knight can move from (i,j) to any eight squares $(i \pm 1, j \pm 2)$ or $(i \pm 2, j \pm 1)$, provided of course that they are on the chessboard. Let X_n be the sequence of squares that results if we pick one of knight's legal moves at random. Find **(a)** the stationary distribution and **(b)** the expected number of moves to return to corner $(1,1)$ when we start there.

Hint. Apply the results in Example 5.6: Random walks on graphs.

5. Consider a Markov chain on the set $S = \{0, 1, 2, \dots, N\}$ with transition probabilities

$$p(i, i-1) = p(i, i+1) = \frac{1}{2} \quad \text{for} \quad 1 \le i \le N-1, \qquad \text{and} \quad p(0,0) = p(N,N) = 1.$$

Denote $T = \min\{n \ge 0 : X_n = 0 \text{ or } N\}$. Find the expectation

$$h(i) = E\big(X_0 + X_1 + \cdots + X_{T-1} \,|\, X_0 = i\big) \quad \text{for} \quad 1 \le i \le N-1.$$