Math 5652: Introduction to Stochastic Processes: Spring 2008

Homework Assignment 4 (due on Tuesday, March 25).

60 points are distributed between 4 problems, 15 points each. Problems 1 and 2 are taken from the textbook by R. Durrett (Exercises 9.32 and 9.48 in Section 1.9).

1. *Ehrenfest chain.* Consider the Ehrenfest chain with thansition probability

$$p(i,i+1) = (N-i)/N, \quad \text{and} \quad p(i,i-1) = i/N \quad \text{for} \quad 0 \le i \le N.$$

Let $\mu_n = E_x(X_n)$. **(a)** Show that $\mu_{n+1} = 1 + (1-2/N)\mu_n$. **(b)** Use this and induction to conclude that

$$\mu_n = \frac{N}{2} + \Big(1 - \frac{2}{N}\Big)^n (x - N/2).$$

2. Consider the Markov chain with state space $S = \{0,1,2,\ldots\}$ and transition probability

$$p(m,m+1) = \frac{1}{2}\Big(1 - \frac{1}{m+2}\Big) \quad \text{for} \quad m \ge 0$$
$$p(m,m-1) = \frac{1}{2}\Big(1 + \frac{1}{m+2}\Big) \quad \text{for} \quad m \ge 1$$

and $p(0,0) = 1 - p(0,1) = 3/4$. Find the stationary distribution π.

3. Consider the Markov chain with state space $S = \{0,1,2,\ldots\}$ and transition probability

$$p(x,x+1) = p(x,x+2) = \frac{1}{2}, \qquad x \ge 0.$$

Find

$$E_0(T_N), \quad \text{where} \quad T_N = \min\{n \ge 0 : \, X_n \ge N\}, \quad N \ge 1.$$

4. Show that for arbitrarily small $\varepsilon \in (0,1)$, there is a random variable Y with distribution

$$a_k = P(Y = k), \quad k = 0,1,2,\ldots, \quad \sum_{k=0}^{\infty} a_k = 1, \quad a_0 > 0,$$

such that $\mu = E(Y) = 1$, $E(|Y|^{2-\varepsilon}) < \infty$, and the corresponding branching process starting from $X_0 = 1$ satisfies

$$E_1(T_0) < \infty, \quad \text{where} \quad T_0 = \inf\{n \ge 0 : \, X_n = 0\}.$$

Hint. Try Y with $\varphi(t) = E(t^Y)$ satisfying

$$\varphi''(t) = c \cdot (1-t)^{-\alpha} = c \cdot \Big[1 + \frac{\alpha t}{1!} + \frac{\alpha(\alpha+1)t^2}{2!} + \cdots\Big],$$

where $0 < \alpha < \varepsilon$, and $c = \mathit{const} > 0$. Use one of the following two facts:

(I) If $c_n > 0$ and $\dfrac{c_{n+1}}{c_n} \le 1 - \dfrac{\gamma}{n}$ for all $n \ge 1$ with $\gamma = \text{const} > 1$, then $\displaystyle\sum_{n=0}^{\infty} c_n < \infty$.

(II) If $c_n > 0$ and $\dfrac{c_{n+1}}{c_n} \ge 1 - \dfrac{\gamma}{n}$ for all $n \ge 1$ with $\gamma = \text{const} < 1$, then $\displaystyle\sum_{n=0}^{\infty} c_n = \infty$.