Math 8583, Fall 2001

Problems for Final Exam (due on Friday, December 21, till 3:30 pm)

The exam consists of 5 problems, 10 points each. The adjusted score for the course is S=(sum of 2 best HW)*3+(Midterm)*5+(Final)*9.

1. Let functions $a_{ij} = a_{ij}(x,t)$, $i,j = 1,2,\cdots,n$ be defined for $x \in \mathbb{R}^n, t > 0$, and let

$$a_{ij} = a_{ji}, \quad \sum_{i,j=1}^{n} a_{ij}^2 \le \nu^{-2}, \quad \text{and} \quad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge \nu|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n \tag{1}$$

with a constant $\nu \in (0,1]$. Consider the function

$$K_{\alpha,\beta}(x,t) = t^{-\alpha} e^{-\frac{|x|^2}{\beta t}}, \quad x \in \mathbb{R}^n, \ t > 0.$$

Show that there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$, depending only on n and ν, such that for all $x \in \mathbb{R}^n$, $t > 0$,

$$LK_{\alpha_1,\beta_1}(x,t) := \left(\frac{\partial}{\partial t} - \sum_{i,j=1}^{n} a_{ij} D_{ij} \right) K_{\alpha_1,\beta_1}(x,t) \ge 0, \quad LK_{\alpha_2,\beta_2}(x,t) \le 0.$$

2. Use the previous result to show that the problem

$$Lu = 0 \quad \text{in} \quad H_T := \mathbb{R}^n \times (0,T), \quad u(x,0) \equiv 0$$

has at most one solution in the class of functions $u \in C^{2,1}(H_T) \cap C\left(\overline{H_T}\right)$, satisfying the inequality $|u(x,t)| \le N \exp\left(a|x|^2\right)$ in H_T with some positive constants N and a.

3. Let Ω be a connected open set in $\mathbb{R}^n$, and let $u \in C^2(\Omega)$ satisfy

$$Lu := -\sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} b_i D_i u + cu \le 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad \sup_{\Omega} u = u(x_0) = 0, \quad x_0 \in \Omega,$$

where a_{ij} satisfy (1) and b_i, c are bounded in Ω. Show that $u \equiv 0$ in Ω.

4. Show that there are no strictly positive solutions of the equation

$$\Delta u = u^2 \qquad \text{on} \quad \mathbb{R}^n.$$

Hints. Step 1 (3 points). Show that the function

$$v_0(x) = \left(1 - |x|^2\right)^{-2} \qquad \text{satisfies} \qquad \Delta v_0 < N_0 v_0^2 \quad \text{in} \quad B_1 = \{|x| < 1\}, \quad \text{where} \quad N_0 = N_0(n).$$

Step 2 (2 points). For some positive constants r and ε, the function

$$v(x) = \varepsilon v_0\left(r^{-1}x\right) \qquad \text{satisfies} \qquad \Delta v < v^2 \quad \text{in} \quad B_r = \{|x| < r\}.$$

Step 3 (5 points). Do the rest.

(over)

5. Let Ω be a bounded open set in $\mathbb{R}^n$, such that for any $x_0 \in \partial\Omega$, the ball $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ contains a smaller ball $B_{\lambda r}(y_0)$ of radius λr, which does not intersect Ω, i.e. $B_{\lambda r}(y_0) \subset B_r(x_0) \setminus \Omega$. Here $r \in (0,1]$ and $\lambda \in (0, 1/2]$ are fixed constants not depending on $x_0 \in \partial\Omega$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to the problem

$$Lu = -\sum_{i,j=1}^{n} a_{ij} D_{ij} u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

where a_{ij} satisfy (1), and f is such that

$$f \equiv 0 \quad \text{in} \quad \Omega^r := \{x \in \Omega : \, dist\,(x, \partial\Omega) > r\}, \quad |f| \le 1 \quad \text{on} \quad \Omega \setminus \Omega^r.$$

Show that $|u| \le N r^2$, with a constant N depending only on n, ν, λ.

(Partial credit of 5 points for solution in a particular case $\Omega = B_1(0)$.)

For solving this problem, one can use the following

Lemma. *Let coefficients of operator* $L := -\sum a_{ij} D_{ij}$ *be defined and satisfy* (1) *in an open set* $\Omega \subset R^n$, *and* $x_0, y_0 \in R^n \setminus \Omega$ *and constants* $r > 0, \lambda \in (0, 1/2]$ *be such that* $B_{\lambda r}(y_0) \subset B_r(x_0) \setminus \Omega$. *Then for arbitrary function* $v \in C^2(\Omega) \cap C(\overline{\Omega})$ *such that*

$$Lv = -\sum_{i,j=1}^{n} a_{ij} D_{ij} v \le 0 \quad \text{in} \quad \Omega \cap B_{4r}(x_0), \quad v \le 0 \quad \text{on} \quad (\partial\Omega) \cap B_{4r}(x_0),$$

we have

$$\sup_{\Omega \cap B_r(x_0)} v^+ \le \theta \cdot \sup_{\Omega \cap B_{4r}(x_0)} v^+,$$

where $v^+ = \max(v, 0)$, *and* $\theta \in (0,1)$ *is a constant depending only on* n, ν *and* λ.

Proof *of Lemma.* Without loss of generality, we may assume

$$r = 1, \quad y_0 = 0, \quad \text{and} \quad M := \sup_{\Omega \cap B_4(x_0)} v > 0.$$

From our assumptions it follows

$$B_\lambda(0) \subset B_1(x_0) \setminus \Omega \subset B_1(x_0) \subset B_2(0) \subset B_3(0) \subset B_4(x_0).$$

Choose $m = m(n, \nu) > 0$ such that $L\left(|x|^{-m}\right) \le 0$ for $x \neq 0$, and consider the function

$$w := v - \frac{M}{\lambda^{-m} - 3^{-m}} \left(\lambda^{-m} - |x|^{-m}\right) \quad \text{in} \quad \Omega' := \Omega \cap B_3(0) \subset \Omega \cap B_4(x_0).$$

It is easy to see that

$$Lw \le 0 \quad \text{in} \quad \Omega', \quad v \le 0 \quad \text{on} \quad \partial\Omega'.$$

By the maximum principle, $w \le 0$ in Ω'. Therefore,

$$v \le \frac{M}{\lambda^{-m} - 3^{-m}} \left(\lambda^{-m} - |x|^{-m}\right) \le \theta M \quad \text{in} \quad (\Omega \cap B_1(x_0)) \subset (\Omega \cap B_2(0))$$

with

$$\theta := \frac{\lambda^{-m} - 2^{-m}}{\lambda^{-m} - 3^{-m}} < 1.$$

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