Math 8583: Theory of Partial Differential Equations: Fall 2001
Midterm Exam: Problems and Solutions

November 14, 2001

1. Let both functions u and u^2 be harmonic in an open set $\Omega \subset \mathbb{R}^n$.
Check whether or not these assumptions imply that (a) e^u is harmonic in Ω; (b) $u \equiv const$ in Ω.

Solution. We have $D\left(u^2\right) = 2uDu,\ \Delta\left(u^2\right) = 2\left|Du\right|^2 + 2u\Delta u = 0,\ \Delta u = 0$, therefore $Du \equiv 0$ in Ω, and $u = const$ on each connected component of Ω. Hence (a) is true for any open set $\Omega \subset \mathbb{R}^n$, and (b) is true for open **connected** sets $\Omega \subset \mathbb{R}^n$. $\square$

2. Denote

$$K_a(x) = \frac{a}{\pi\left(x^2+a^2\right)} \quad \text{for} \quad a > 0,\ x \in \mathbb{R}^1.$$

Evaluate the convolution $K_a * K_b(x)$ for $a > 0, b > 0$.

Solution. For $g \in \left(C \cap L^\infty\right)\left(\mathbb{R}^1\right)$, the function $u(x,y) := g * K_y(x)$ is a solution to the problem

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} = 0 \quad \text{in} \quad \mathbb{R}^2_+ := \left\{(x,y) \in \mathbb{R}^2 :\ y > 0\right\}, \\ u(x,y) &\to g(x_0) \quad \text{as} \quad x \to x_0,\ y \to 0^+. \end{aligned}$$

On the other hand, the function $v(x,y) := u(x, y+b)$ is a solution to a similar problem with $u(x,b)$ instead of $g(x)$. Therefore, we can write

$$g * K_{a+b}(x) = u(x, a+b) = v(x,a) = u(\cdot, b) * K_a(x) = (g * K_b) * K_a(x) = g * (K_a * K_b)(x).$$

Choosing functions $g_\varepsilon \in C_0^\infty(-\varepsilon, \varepsilon)$ satisfying $g_\varepsilon \geq 0$ and $\int g_\varepsilon dx = 1$, we conclude

$$(K_a * K_b)(x) = \lim_{\varepsilon \to 0+} g_\varepsilon * (K_a * K_b)(x) = \lim_{\varepsilon \to 0+} g_\varepsilon * K_{a+b}(x) = K_{a+b}(x).$$

$\square$

3. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx \quad \text{for} \quad a > 0$$

using the previous hint and the fact that $u(x,y) := \cos x \cdot e^{-y}$ satisfies $\Delta u = u_{xx} + u_{yy} = 0$.

Solution. We have

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx = \frac{\pi}{a} \int_{-\infty}^{\infty} \cos x \cdot K_a(-x)\ dx = \frac{\pi}{a}(\cos x) * K_a(0) = \frac{\pi}{a} u(0,a) = \frac{\pi}{a} e^{-a}.$$

$\square$

4. Let $u \in C^\infty(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ where $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}$, and let

$$\Delta u(x) \equiv 0, \quad |u(x)| \le 1 + |x|^\alpha \quad \text{in} \quad \mathbb{R}^n_+, \quad \text{and} \quad u \equiv 0 \quad \text{on} \quad \partial\mathbb{R}^n_+,$$

where $\alpha = const \in (0,1)$. Show that $u \equiv 0$ in $\mathbb{R}^n_+$. Check whether or not this statement is true in the case $\alpha = 1$.

Solution. A simple example $u(x) \equiv x_n$ shows that this statement is not true for $\alpha = 1$.

For $0 < \alpha < 1$, we will use the odd continuation argument. Since u is a harmonic function which vanishes on a flat boundary, by the odd continuation

$$u(x', -x_n) \equiv -u(x', x_n) \quad \text{for} \quad x' \in \mathbb{R}^{n-1}, x_n > 0$$

we get a harmonic function on the whole space $\mathbb{R}^n$. For an arbitrary $x \in \mathbb{R}^n$ and $r > 2|x|$, we have

$$|Du(x)| \le \sup_{B_{r/2}} |Du| \le \frac{N}{r} \sup_{B_r} |u| \le \frac{N}{r}(1 + r^\alpha) \to 0 \quad \text{as} \quad r \to +\infty.$$

Here $B_r := \{x \in \mathbb{R}^n; \ |x| < r\}$ and N is a constant depending only on n. The above relation implies

$$Du \equiv 0, \quad \text{and} \quad u \equiv const = 0 \quad \text{on} \quad \mathbb{R}^n.$$

□

5. Let $u \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$ be a solution to the problem

$$\Delta u = -1 \quad \text{in} \quad \Omega := \left\{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1, \ |x_2| < 1\right\}, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

Show that

$$\sup_\Omega u = u(0).$$

Proof. Together with $u = u(x_1, x_2)$, the functions $u(-x_1, x_2)$ and $u(x_1, -x_2)$ are solutions of this problem. By uniqueness,

$$u(x_1, x_2) \equiv u(-x_1, x_2) \equiv u(x_1, -x_2) \quad \text{in} \quad \Omega,$$

and therefore $v_1 := D_1 u = 0$ for $x_1 = 0$, $v_2 := D_2 u = 0$ for $x_2 = 0$. The functions $v_1, v_2 \in C^\infty(\Omega) \cap C(\overline{\Omega})$ and satisfy

$$\Delta v_k = \Delta D_k u = D_k \Delta u = D_k(-1) = 0 \quad \text{in} \quad \Omega.$$

By the maximum (or comparison) principle $u \ge 0$ in Ω. Since $u(-1, x_2) \equiv 0$, this implies $v_1 = D_1 u \ge 0$ on $(\partial\Omega) \cap \{x_1 = -1\}$. Moreover, from $u(x_1, \pm 1) \equiv 0$ it follows $v_1 \equiv 0$ on $(\partial\Omega) \cap \{x_2 = \pm 1\}$. Hence v_1 is harmonic in the region $\Omega_1 := \Omega \cap \{x_1 < 0\}$ and $v_1 \ge 0$ on $\partial\Omega_1$. By the maximum principle, $v_1 = D_1 u \ge 0$ in Ω_1. Taking into account $u(x_1, x_2) \equiv u(-x_1, x_2)$, we get

$$\max_{|x_1| \le 1} u(x_1, x_2) = u(0, x_2) \quad \text{for} \quad |x_2| < 1.$$

Quite similarly,

$$\max_{|x_2| \le 1} u(x_1, x_2) = u(x_1, 0) \quad \text{for} \quad |x_1| < 1,$$

and the desired equality $\sup u = u(0)$ follows. □