Math 8583: Theory of Partial Differential Equations: Fall 2001

Homework Assignment 2 (due on Wednesday, November 7, till 12:05 pm)

50 points are distributed between 5 problems, 10 points each.

1. Let $u = u(t, x)$ be a solution of the heat equation $u_t = u_{xx}$. Consider the function

$$v = v(s, y) = \sqrt{t}u(t, x) \exp\left(\frac{x^2}{4t}\right), \qquad \text{where} \quad s = -\frac{1}{t}, \ y = \frac{x}{t}.$$

Evaluate

$$\frac{\partial v}{\partial s} - \frac{\partial^2 v}{\partial y^2}.$$

2. Let $g(x)$ be a bounded continuous function on $\mathbb{R}^n$, such that

$$\int_{\mathbb{R}^n} g(x)\, dx > 0, \quad \text{and} \quad g(x) \equiv 0 \quad \text{for} \quad |x| \geq 1, \tag{1}$$

and let $u(x, t)$ be a bounded solution to the Cauchy problem

$$u_t = \Delta_x u \quad \text{for} \quad t > 0, \quad u(x, 0) \equiv g(x). \tag{2}$$

Show that for arbitrary $R > 0$, there exists $T > 0$ such that

$$u(x, t) > 0 \quad \text{for all } (x, t) \text{ satisfying} \quad |x| \leq R, \ t \geq T.$$

3. In the case $n = 1$, check whether or not the above assumptions (1) and (2) guarantee the existence of $T > 0$ such that

$$u(x, t) > 0 \quad \text{for all} \quad x \in \mathbb{R}^1, \ t \geq T.$$

4. Let $u(x) = u(x_1, x_2)$ be a bounded solution of the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad \mathbb{R}^2_+ = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\right\} \tag{3}$$

with the boundary condition

$$u(x, 0) \equiv g(x) = \frac{|x|}{1 + x^2}.$$

Show that the gradient Du is unbounded on $\mathbb{R}^2_+$. You can use the explicit expression for this solution:

$$u(x_1, x_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_2 g(t) dt}{(x_1 - t)^2 + x_2^2}.$$

5. Let $u(x) = u(x_1, x_2)$ be a bounded solution of the Laplace equation (3) with the boundary condition $u(x_1, 0) \equiv g(x_1)$, where g is a bounded continuous function on $\mathbb{R}^1$ satisfying

$$[g]_\alpha := \sup\left\{\frac{|g(t) - g(s)|}{|t - s|^\alpha} : \quad t, s \in \mathbb{R}^1, \ t \neq s\right\}$$

with a constant $\alpha \in (0, 1)$. Show that

$$[u]_\alpha := \sup\left\{\frac{|u(x) - u(y)|}{|x - y|^\alpha} : \quad x, y \in \mathbb{R}^2_+, \ x \neq y\right\} \leq N \cdot [g]_\alpha$$

with a constant N depending only on n and α.