Math 8583: Theory of Partial Differential Equations: Fall 2001

Homework Assignment 2. Problems and Solutions

1. Let $u = u(t,x)$ be a solution of the heat equation $u_t = u_{xx}$. Consider the function

$$v = v(s,y) = \sqrt{t}u(t,x)\exp\left(\frac{x^2}{4t}\right), \qquad \text{where} \quad s = -\frac{1}{t},\ y = \frac{x}{t}.$$

Evaluate

$$\frac{\partial v}{\partial s} - \frac{\partial^2 v}{\partial y^2}.$$

Solution. We can write $u(t,x) = K(t,x)v(s,y)$, where $K(t,x) = t^{-1/2}\exp\left(-\frac{x^2}{4t}\right)$. Since $K_t - K_{xx} = 0$, we have

$$\begin{aligned} 0 &= u_t - u_{xx} = (K_t - K_{xx})\,v - 2K_x \cdot v_x + K \cdot (v_t - v_{xx}) \\ &= \frac{x}{t}K \cdot \frac{1}{t}v_y + K \cdot \left(-\frac{x}{t^2}v_y + \frac{1}{t^2}v_s - \frac{1}{t^2}v_{yy}\right) = \frac{1}{t^2}K \cdot (v_s - v_{yy}), \end{aligned}$$

and $v_s - v_{yy} = 0$. □

2. Let $g(x)$ be a bounded continuous function on $\mathbb{R}^n$, such that

$$\int_{\mathbb{R}^n} g(x)\,dx > 0, \quad \text{and} \quad g(x) \equiv 0 \quad \text{for} \quad |x| \ge 1, \tag{1}$$

and let $u(x,t)$ be a bounded solution to the Cauchy problem

$$u_t = \Delta_x u \quad \text{for} \quad t > 0, \quad u(x,0) \equiv g(x). \tag{2}$$

Show that for arbitrary $R > 0$, there exists $T > 0$ such that

$$u(x,t) > 0 \quad \text{for all } (x,t) \text{ satisfying} \quad |x| \le R,\ t \ge T.$$

Proof. We know that

$$u(x,t) = \int_{\mathbb{R}^n} g(y)\,\Phi(x-y,t)\,dy, \quad \text{where} \quad \Phi(x,t) = (4\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right) \quad \text{for} \quad t > 0.$$

Fix $\varepsilon > 0$ satisfying $\varepsilon \int |g(y)|\,dy < \int g(y)\,dy$ and choose $T = T(\varepsilon, R) > 0$ such that

$$\exp\left(-\frac{(R+1)^2}{4T}\right) \ge 1 - \varepsilon.$$

Then

$$1 - \varepsilon \le \exp\left(-\frac{|x|^2}{4t}\right) \le 1 \quad \text{for} \quad |x| \le R+1, \quad t \ge T.$$

For $|x| \le R,\ t \ge T$, we have

$$\begin{aligned}(4\pi t)^{n/2} u(x,t) &= \int\limits_{|y|\le 1} g(y)\exp\left(-\frac{|x-y|^2}{4t}\right) dy \\ &= \int g(y)\, dy - \int\limits_{|y|\le 1} g(y)\left[1-\exp\left(-\frac{|x-y|^2}{4t}\right)\right] dy \\ &\ge \int g(y)\, dy - \varepsilon \int |g(y)|\, dy > 0,\end{aligned}$$

so that $u(x,t) > 0$. □

3. In the case $n = 1$, check whether or not the above assumptions (1) and (2) guarantee the existence of $T > 0$ such that

$$u(x,t) > 0 \quad \text{for all} \quad x \in \mathbb{R}^1,\ t \ge T.$$

Solution. Choose a function $g \in C(\mathbb{R}^1)$ such that

$$g(x) \equiv 0 \quad \text{for} \quad |x| \ge 1, \quad \int\limits_{-\infty}^{\infty} g(x)\, dx > 0, \quad g > 0 \quad \text{in} \quad (-1,0), \quad g < 0 \quad \text{in} \quad (0,1).$$

We claim that for arbitrary $t > 0$, the inequality $u(x,t) < 0$ holds for all large enough $x > 0$, and therefore there is no constant $T > 0$ with given property. For the proof of this claim, fix an arbitrary constant $a \in (0,1)$ and denote

$$M := \int\limits_0^1 g(y)\, dy > 0, \quad m := -\int\limits_a^1 g(y)\, dy > 0.$$

Then for $x > 1, t > 0$,

$$\begin{aligned}\sqrt{4\pi t}\cdot u(x,t) &= \int\limits_{-1}^{1} g(y)\exp\left[-\frac{(x-y)^2}{4t}\right] dy \\ &< \int\limits_{-1}^{0} g(y)\exp\left(-\frac{x^2}{4t}\right) dy + \int\limits_a^1 g(y)\exp\left[-\frac{(x-a)^2}{4t}\right] dy \\ &= \exp\left(-\frac{x^2}{4t}\right)\cdot\left[M - m\exp\left(\frac{2ax-a^2}{4t}\right)\right].\end{aligned}$$

Obviously, the last expression is negative for large $x > 0$. □

4. Let $u(x) = u(x_1,x_2)$ be a bounded solution of the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad \mathbb{R}^2_+ = \left\{x = (x_1,x_2) \in \mathbb{R}^2 : x_2 > 0\right\} \tag{3}$$

with the boundary condition

$$u(x,0) \equiv g(x) = \frac{|x|}{1+x^2}.$$

Show that the gradient Du is unbounded on $\mathbb{R}^2_+$. You can use the explicit expression for this solution:

$$u(x_1, x_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_2 g(t)dt}{(x_1 - t)^2 + x_2^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_2 g(x_1 - t)dt}{t^2 + x_2^2}. \tag{4}$$

Proof. By the monotone convergence theorem

$$\frac{u(0,y) - u(0,0)}{y} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|t|\, dt}{(t^2 + y^2)(1 + t^2)} \to \infty \quad \text{as} \quad y \to 0+,$$

because

$$\frac{|t|}{(t^2 + y^2)(1 + t^2)} \nearrow \frac{1}{|t|(1 + t^2)} \quad \text{as} \quad y \to 0+,$$

and the limit function is not integrable. On the other hand,

$$\frac{u(0,y) - u(0,0)}{y} = \frac{1}{y} \int_0^1 \frac{d}{dt} u(0, ty)\; dt = \int_0^1 D_y u(0, ty)\; dt,$$

and hence $D_y(0,y)$ is unbounded for small $y > 0$. □

5. Let $u(x) = u(x_1, x_2)$ be a bounded solution of the Laplace equation (3) with the boundary condition $u(x_1, 0) \equiv g(x_1)$, where g is a bounded continuous function on $\mathbb{R}^1$ satisfying

$$[g]_\alpha := \sup \left\{ \frac{|g(t) - g(s)|}{|t - s|^\alpha} : \quad t, s \in \mathbb{R}^1,\ t \neq s \right\}$$

with a constant $\alpha \in (0,1)$. Show that

$$[u]_\alpha := \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} : \quad x, y \in \mathbb{R}^2_+,\ x \neq y \right\} \leq N \cdot [g]_\alpha$$

with a constant N depending only on n and α.

Proof. For $x, y \in \mathbb{R}^2_+$, we have

$$|u(x) - u(y)| = |u(x_1, x_2) - u(y_1, y_2)| \leq |u(x_1, x_2) - u(y_1, x_2)| + |u(y_1, x_2) - u(y_1, y_2)|. \tag{5}$$

From the representation (4) it follows

$$|u(x_1, x_2) - u(y_1, x_2)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_2 |g(x_1 - t) - g(y_1 - t)|\, dt}{t^2 + x_2^2}.$$

Denote $A := [g]_\alpha$. Since $|g(x_1 - t) - g(y_1 - t)| \leq A|x_1 - y_1|^\alpha \leq A|x - y|^\alpha$, we get

$$|u(x_1, x_2) - u(y_1, x_2)| \leq \frac{A|x - y|^\alpha}{\pi} \int_{-\infty}^{\infty} \frac{x_2 dt}{t^2 + x_2^2} = A|x - y|^\alpha. \tag{6}$$

Further, using substitution $t = y_1 + s y_2$, we can write

$$u(y_1, y_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y_2 g(t)dt}{(y_1 - t)^2 + y_2^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(y_1 + s y_2)\, ds}{s^2 + 1},$$

hence

$$|u(y_1, x_2) - u(y_1, y_2)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|g(y_1 + sx_2) - g(y_1 + sy_2)| \, ds}{s^2 + 1}.$$

Since $|g(y_1 + sx_2) - g(y_1 + sy_2)| \leq A |sx_2 - sy_2|^\alpha \leq A |s|^\alpha |x - y|^\alpha$, we get

$$|u(y_1, x_2) - u(y_1, y_2)| \leq \frac{A |x - y|^\alpha}{\pi} \int_{-\infty}^{\infty} \frac{|s|^\alpha \, ds}{s^2 + 1} = N_0 A |x - y|^\alpha.$$

Together with (5) and (6), this gives us $|u(x) - u(y)| \leq (1 + N_0) A |x - y|^\alpha$ for all $x, y \in \mathbb{R}^2_+$. This means the estimate $[u]_\alpha \leq N [g]_\alpha$ holds with $N = 1 + N_0$. □