Math 8583, Fall 2003

Problems for Final Exam (revised version, with comments)
(due on Thursday, December 18, till 10:00 am)

The exam consists of 5 problems, 10 points each. The adjusted score for the course is

$$S = (1\text{ best HW, out of 2}) * 5 + (\text{Midterm}) * 6 + (\text{Final}) * 9, \quad S_{\max} = 1000.$$

1 (Kelvin transformation). Let u be a harmonic function in an open set $\Omega \subset \mathbb{R}^n$, $n \ge 1$. Then the function

$$u^*(x) := |x|^{2-n} u(|x|^{-2}x) \quad \text{is harmonic in} \quad \Omega^* := \{x \in \mathbb{R}^n : \; |x|^{-2}x \in \Omega\}.$$

Hint: One can make calculations a bit shorter using the fact that the function $v(x) := |x|^{2-n}$ is harmonic for $x \ne 0$.

2 (Removable singularity). Let $u = u(x)$ be a bounded harmonic function in the punctured disk $B_1 \setminus \{0\} = \{x = (x_1, x_2) \in \mathbb{R}^2 : \; 0 < |x| < 1\}$. Show that one can define $u(0)$ in such a way that $u(x)$ becomes harmonic in the whole disk $B_1 = \{x = (x_1, x_2) \in \mathbb{R}^2 : \; |x| < 1\}$.

Hint: Fix $r \in (0,1)$ and consider the function $v \in C^2_{loc}(B_r) \cap C(\bar{B}_r)$ satisfying

$$\Delta v = 0 \quad \text{in} \quad B_r := \{|x| < r\}, \quad v = u \quad \text{on} \quad \partial B_r.$$

It suffices to check that $u \equiv v$ in $B_r \setminus \{0\}$, so that by defining $u(0) = v(0)$ we get a harmonic function in B_1. Compare $v(x)$ with functions $u(x) \pm \varepsilon \cdot (\ln|x| - \ln r)$, which are harmonic for $x \ne 0$.

3. Let Ω_1 and Ω_2 be bounded open sets in $\mathbb{R}^n$, such that $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and for $k = 1$ and 2, let $u_k \in C^2_{loc}(\Omega_k) \cap C(\bar{\Omega}_k)$ be such that

$$u_k > 0, \quad Lu_k := \sum_{i,j=1}^n a_{ij} D_{ij} u_k = \lambda_k u_k \quad \text{in} \quad \Omega_k, \quad u_k = 0 \quad \text{on} \quad \partial\Omega_k,$$

where $\lambda_k = const$, and the coefficients $a_{ij} = a_{ij}(x)$ satisfy

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \le \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \le \nu^{-1}|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n,$$

with a constant $\nu \in (0,1]$. Show that $\lambda_1 < \lambda_2 < 0$.

Hint: Consider the operators $L_k := L - \lambda_k$, $k = 1$ and 2. Assuming $\lambda_2 \le \lambda_1$, we then have

$$u_2 \ge m = const > 0, \quad L_1 u_2 = L_2 u_2 + (\lambda_2 - \lambda_1) u_2 \le 0 \quad \text{in} \quad \Omega_1.$$

Then proceed as in the proof of Theorem 3.2, Chapter 0, with

$$\Omega = \Omega_1, \quad L = L_1, \quad u = u_1, \quad w = u_2.$$

4. Show that the problem

$$\Delta u = u^2 \quad \text{in} \quad B_1 = \{|x| < 1\} \subset \mathbb{R}^n, \quad u(x) \to +\infty \quad \text{as} \quad |x| \to 1^-,$$

cannot have more than one non-negative solution in $C^2_{loc}(B_1)$.

Hint. Step 1. Show that the following comparison principle holds for the nonlinear operator $Tu := \Delta u - u^2$: if $Tu_1 \geq Tu_2$ in a bounded open set Ω, and $0 \leq u_1 \leq u_2$ on $\partial\Omega$, then $u_1 \leq u_2$ in Ω.

Step 2. Note that if the function $u(x)$ is a solution to the given problem, then for any constant $r > 0$, there exists a constant $c_r > 0$ such that the function

$$u_r(x) := c_r \cdot u(r^{-1}x) \quad \text{satisfies} \quad \Delta u_r = u_r^2 \quad \text{in} \quad B_r := \{|x| < r\}.$$

5. Show that the *Hermite functions*

$$h_k(x) := e^{-\frac{x^2}{2}} H_k(x), \quad \text{where} \quad H_k(x) := (-1)^k e^{x^2} \left(e^{-x^2}\right)^{(k)}, \quad x \in \mathbb{R}^1,$$

are eigenfunctions of the Fourier transform, i.e. $\mathcal{F}[h_k](\xi) = c_k h_k(\xi), \quad k = 0, 1, 2, \dots.$

Hint. Step 1. Show that $\mathcal{F}[h_k](\xi) = e^{-\frac{\xi^2}{2}} P_k(\xi)$, where P_k is a polynomial of degree k.

Step 2. Note that the statement is true for $k = 0$ (formula (6.11_a) in Lecture Notes). Proceed by induction with respect to k, using the orthogonality of functions h_k in L^2 :

$$\int h_j h_k \, dx = 0 \quad \text{for} \quad j \neq k,$$

and the equality (6.8):

$$\int f_1 \cdot \mathcal{F}[f_2] \, dx = \int f_2 \cdot \mathcal{F}[f_1] \, dx \quad \text{for} \quad f_{1,2} \in L^1.$$