Math 8583, Fall 2003

Solutions for Final Exam

1 (Kelvin transformation). Let u be a harmonic function in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Then the function

$$u^*(x) := |x|^{2-n} u(|x|^{-2}x) \quad \text{is harmonic in} \quad \Omega^* := \{x \in \mathbb{R}^n : \ |x|^{-2}x \in \Omega\}.$$

Proof. This statement is a special case, with $R = 1$, of the following one: for any constant $R > 0$, the function

$$u^*(x) := \left(\frac{R}{|x|}\right)^{n-2} u(x^*) \quad \text{is harmonic in} \quad \Omega^* := \left\{x \in \mathbb{R}^n \setminus \{0\} : \ x^* := \frac{R^2}{|x|^2} x \in \Omega\right\}.$$

We will avoid direct calculations by using the *fundamental solution* for the operator $-\Delta$, which is defined for $x \neq 0$ as follows:

$$\Gamma(x) = \left\{ \begin{array}{ll} c_n |x|^{2-n} & \text{if} \quad n \neq 2, \\ -\frac{1}{2\pi} \ln |x| & \text{if} \quad n = 2, \end{array} \right. \qquad \text{where} \quad c_n = \frac{1}{(n-2)\sigma_n}, \ \sigma_n := |\partial B_1| = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{1}$$

In a special case $n = 1$, we have $\sigma_1 = 2$, and $\Gamma(x) = -\frac{1}{2}|x|$ for $x \neq 0$.

Fix an arbitrary point $x_0 \in \Omega^*$and choose a function $\zeta \in C_0^\infty(\Omega \setminus \{0\})$ such that $\zeta \equiv 1$ in a neighborhood of the point $x_0^* := R^2|x_0|^{-2}x_0 \in \Omega \setminus \{0\}$. Since the harmonic function $u \in C_{loc}^\infty(\Omega)$, we also have

$$v := u\zeta \in C_0^\infty(\Omega \setminus \{0\}) \subset C_0^\infty(\mathbb{R}^n) \quad \text{and} \quad f := -\Delta v \in C_0^\infty(\Omega \setminus \{0, x_0^*\}) \subset C_0^\infty(\mathbb{R}^n).$$

By properties of the fundamental solution Γ,

$$v = -\Delta(\Gamma * v)) = \Gamma * (-\Delta v) = \Gamma * f \quad \text{on} \quad \mathbb{R}^n.$$

Choose a small $\varepsilon \in (0, |x_0|)$ such that for any $x \in B_\varepsilon(x_0)$, the corresponding point $x^* = R^2|x|^{-2}x$ belongs to the neighborhood of x_0^*, in which $\zeta \equiv 1, f \equiv 0$. Then for $x \in B_\varepsilon(x_0)$, we have

$$\begin{aligned} u^*(x) &= R^{n-2}|x|^{2-n}u(x^*) = R^{n-2}|x|^{2-n}v(x^*) \\ &= R^{n-2}|x|^{2-n}(\Gamma * f)(x^*) = R^{n-2}|x|^{2-n} \int f(y)\Gamma(x^* - y)\, dy. \end{aligned}$$

Further, we involve some geometrical considerations. Since $|x| \cdot |x^*| = R^2$, the points x and x^* are mutually symmetric (dual) with respect to the sphere ∂B_R, and $(x^*)^* = x$. For arbitrary x and y in $\mathbb{R}^n \setminus \{0\}$, from $|x| \cdot |x^*| = R^2 = |y| \cdot |y^*|$ it follows $|x| : |y^*| = |y| : |x^*|$. Therefore, the triangles $0xy^*$ and $0yx^*$ are similar, and

$$\frac{|x|}{|y|} = \frac{|y^*|}{|x^*|} = \frac{|x - y^*|}{|y - x^*|}.$$

In the case $n \neq 2$, we can write

$$u^*(x) = c_n R^{n-2} \int f(y) \cdot |x|^{2-n} |x^* - y|^{2-n}\, dy = c_n R^{n-2} \int f(y) \cdot |y|^{2-n}\, |x - y^*|^{2-n} dy$$

for $x \in B_\varepsilon(x_0)$. Note that $f(y) \equiv 0$ near 0, and also for y close to x^*, or equivalently, for x close to y^*. Then the integral functions do not have singularities, and

$$\Delta u^*(x) = c_n R^{n-2} \int f(y) \cdot |y|^{2-n} \Delta_x |x - y^*|^{2-n}\, dy = 0$$

for $x \in B_\varepsilon(x_0)$. Since x_0 is an arbitrary point in Ω^*, we have $\Delta u^* \equiv 0$ in Ω^*.

In the remaining case $n = 2$,

$$\begin{aligned} u^*(x) &= -\frac{1}{2\pi}\int f(y)\cdot \ln|x^*-y|\,dy = -\frac{1}{2\pi}\int f(y)\cdot\ln\left(\frac{|x-y^*|\cdot|y|}{|x|}\right)dy \\ &= \frac{1}{2\pi}\int f(y)\cdot(\ln|x|-\ln|x-y^*|-\ln|y|)\,dy, \\ \Delta u^*(x) &= \frac{1}{2\pi}\int f(y)\cdot(\Delta_x\ln|x|-\Delta_x\ln|x-y^*|)\,dy = 0 \end{aligned}$$

for $x \in B_\varepsilon(x_0)$, hence $\Delta u^* \equiv 0$ in Ω^*. □

2 (Removable singularity). Let $u = u(x)$ be a bounded harmonic function in the punctured disk $B_1 \setminus \{0\} = \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < |x| < 1\}$. Show that one can define $u(0)$ in such a way that $u(x)$ becomes harmonic on the whole disk $B_1 = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}$.

Proof. We prove a stronger statement for any dimension $n \geq 1$: *Let u be a harmonic function in the punctured ball $B_1 \setminus \{0\} = \{x \in \mathbb{R}^n : 0 < |x| < 1\}$, such that $u(x) = o(\Gamma(x))$ as $x \to 0$, i.e.*

$$u(x)/\Gamma(x) \to 0 \quad \text{as} \quad x \to 0, \tag{2}$$

where $\Gamma(x)$ is defined in (1). Then the function u can be extended as a harmonic function to the whole ball B_1.

The case $n = 1$ is not interesting, because from our assumptions it follows

$$u''(x) \equiv 0 \quad \text{for} \quad 0 < |x| < 1, \quad \text{and} \quad u(x)/x \to 0 \quad \text{as} \quad x \to 0,$$

which is possible only in the trivial case $u \equiv 0$ in $(-1, 1)$. Therefore, we restrict ourselves to $n \geq 2$. Then

$$\Gamma > 0 \quad \text{in} \quad B_1 \setminus \{0\}, \quad \text{and} \quad \Gamma(x) \to +\infty \quad \text{as} \quad x \to 0.$$

Fix $r \in (0, 1)$ and consider the solution $v \in C^\infty_{loc}(B_r) \cap C(\bar{B}_r)$ to the problem

$$\Delta v = 0 \quad \text{in} \quad B_r := \{|x| < r\}, \quad v = u \quad \text{on} \quad \partial B_r.$$

For arbitrary $\varepsilon > 0$, one can choose $\delta \in (0, r)$ such that $|u - v| \leq \varepsilon\Gamma$ on $\bar{B}_\delta \setminus \{0\}$. Then

$$v_1 := v - \varepsilon\Gamma \leq u \leq v_2 := v + \varepsilon\Gamma \quad \text{on} \quad \partial(B_r \setminus B_\delta) = (\partial B_r) \cup (\partial B_\delta).$$

Applying the comparison principle to the harmonic functions v_1, v_2 and u, we have

$$v_1 \leq u \leq v_2, \quad \text{i.e.} \quad |u - v| \leq \varepsilon\Gamma \quad \text{in} \quad B_r \setminus B_\delta.$$

Letting $\delta \to 0^+$, and then $\varepsilon \to 0^+$, we finally get $u \equiv v$ on $B_r \setminus \{0\}$, so that $u(0) = v(0)$ is the desired extension of u to B_1. □

3. Let Ω_1 and Ω_2 be bounded open sets in $\mathbb{R}^n$, such that $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and for $k = 1$ and 2, let $u_k \in C^2_{loc}(\Omega_k) \cap C(\bar{\Omega}_k)$ be such that

$$u_k > 0, \quad Lu_k := \sum_{i,j=1}^n a_{ij} D_{ij} u_k = \lambda_k u_k \quad \text{in} \quad \Omega_k, \quad u_k = 0 \quad \text{on} \quad \partial\Omega_k,$$

where $\lambda_k = const$, and the coefficients $a_{ij} = a_{ij}(x)$ satisfy

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n,$$

with a constant $\nu \in (0, 1]$. Show that $\lambda_1 < \lambda_2 < 0$.

Proof. Suppose $\lambda_k \geq 0$. Since $u_k > 0$ in Ω_k, we then have $Lu_k = \lambda_k u_k \geq 0$ in Ω_k. By the maximum principle,

$$\sup_{\Omega_k} u_k = \sup_{\partial\Omega_k} u_k = 0,$$

which contradicts our assumption $u_k > 0$ in Ω_k. Therefore, we must have $\lambda_k < 0$ for $k = 1$ and 2.

Further, since u_2 is strictly positive on $\bar{\Omega}_1$, we have $v := u_1/u_2 \in C^2_{loc}(\Omega_1) \cap C(\bar{\Omega}_1)$, $v > 0$ in Ω_1, and $v = 0$ on $\partial\Omega_1$. Moreover,

$$Lu_1 = L(u_2 v) = u_2 Lv + 2\sum_{i,j=1}^{n} a_{ij} D_i v\, D_j u_2 + Lu_2 \cdot v \quad \text{in} \quad \Omega_1.$$

Having in mind that $Lu_1 = \lambda_1 u_1 = \lambda_1 u_2 v$, $Lu_2 \cdot v = \lambda_2 u_2 v$, we can write

$$\bar{L}v := \sum_{i,j=1}^{n} \bar{a}_{ij} D_{ij} v + \sum_{i=1}^{n} \bar{b}_i D_i v + \bar{c} v = 0 \quad \text{in} \quad \Omega_1, \quad v = 0 \quad \text{on} \quad \partial\Omega_1, \tag{3}$$

where

$$\bar{a}_{ij} = u_2 a_{ij}, \quad \bar{b}_i = \sum_{j=1}^{n} a_{ij} D_j u_2, \quad \bar{c} = (\lambda_2 - \lambda_1) u_2.$$

If $\lambda_1 \geq \lambda_2$, then $\bar{c} \leq 0$, and the problem (3) has a unique solution $v \equiv 0$ in Ω_1. However, we know that $v > 0$ in Ω_1. This contradiction proves the inequalities $\lambda_1 < \lambda_2 < 0$. □

4. Show that the problem

$$\Delta u = u^2 \quad \text{in} \quad B_1 = \{|x| < 1\} \subset \mathbb{R}^n, \quad u(x) \to +\infty \quad \text{as} \quad |x| \to 1^-, \tag{4}$$

cannot have more than one non-negative solution in $C^2_{loc}(B_1)$.

Proof. *Step 1.* For fixed $r > 1$, let $v \in C^2_{loc}(B_r)$ be a non-negative solution to the problem

$$\Delta v = v^2 \quad \text{in} \quad B_r, \quad v(x) \to +\infty \quad \text{as} \quad |x| \to r - 0. \tag{4_r}$$

We claim that for any non-negative solution u to the problem (4), we have $u \geq v$ in B_1. Indeed, if we suppose otherwise, then $\Omega := B_1 \cap \{u < v\}$ is a non-empty open set, $\Omega \subset \bar{\Omega} \subset B_1$,

$$\Delta u = u^2 < v^2 = \Delta v \quad \text{in} \quad \Omega, \quad u = v \quad \text{on} \quad \partial\Omega.$$

Then by the comparison principle, we must have $u \geq v$ in Ω, in contradiction with the definition of Ω. This contradiction proves the desired inequality $u \geq v$ in B_1.

Step 2. Let u_1 and u_2 be solutions to the problem (4). For fixed $r > 1$, define $v(x) := r^{-2} u_1(r^{-1}x)$ in B_r. We have

$$\Delta v(x) = r^{-4}(\Delta u_1)(r^{-1}x) = r^{-4} u_1^2(r^{-1}x) = v^2(x),$$

so that v is a solution to the problem (4_r). Applying the previous step with $u = u_2$, we derive

$$u_2(x) \geq v(x) = r^{-2} u_1(r^{-1}x) \quad \text{for all} \quad x \in B_1 \quad \text{and} \quad r > 1,$$

and

$$u_2(x) \geq \lim_{r \to 1^+} r^{-2} u_1(r^{-1}x) = u_1(x) \quad \text{on} \quad B_1.$$

Interchanging u_1 and u_2, we also get $u_1 \geq u_2$ in B_1. Hence $u_1 \equiv u_2$ on B_1, and the problem (4) cannot have two different solutions. □

5. Show that the *Hermite functions*

$$h_k(x) := e^{-\frac{x^2}{2}} H_k(x), \quad \text{where} \quad H_k(x) := (-1)^k e^{x^2} \left(e^{-x^2}\right)^{(k)}, \quad x \in \mathbb{R}^1,$$

are eigenfunctions of the Fourier transform, i.e. $\mathcal{F}[h_k](\xi) = c_k h_k(\xi), \quad k = 0, 1, 2, \ldots$.

Proof is based on the following properties:

(i) (p. O-40) $h_0 = e^{-\frac{x^2}{2}}$, $\mathcal{F}[h_0] = c_0 h_0$ with $c_0 = \sqrt{2\pi}$.

(ii) (p. O-50) For each $k = 0, 1, 2, \ldots,$ H_h is a polynomial of degree k, and functions $h_k := H_k e^{-\frac{x^2}{2}}$ are orthogonal in $L^2(\mathbb{R}^1) : \int h_j h_k \, dx = 0$ for $j \neq k$.

(iii) (p. O-44) $\mathcal{F}[x^l f] = (iD)^l \mathcal{F}[f]$, $\mathcal{F}[D^\lambda f] = (i\xi)^\lambda \mathcal{F}[f]$.

(iv) (p. O-40) $\int f_1 \cdot \mathcal{F}[f_2] \, dx = \int f_2 \cdot \mathcal{F}[f_1] \, dx$ for $f_{1,2} \in L^1$.

The property (i) gives yields our statement for $k = 0$. Suppose we already have $\mathcal{F}[h_j](\xi) = c_j h_j$ for $j = 0, 1, \ldots, k$ with some constants c_j. By induction, we need to prove a similar equality for $j = k+1$. Let $H_{k+1}(x) = \sum\limits_{l=0}^{k+1} a_l x^l$ with $a_l = const$. Using (i) and (ii), we derive

$$\mathcal{F}[h_{k+1}] = \mathcal{F}[H_{k+1} h_0] = \sum_{l=0}^{k+1} a_l \mathcal{F}[x^l h_0] = \sum_{l=0}^{k+1} a_l (iD)^l \mathcal{F}[h_0] = c_0 \sum_{l=0}^{k+1} a_l (iD)^l h_0 = P_{k+1} h_0,$$

where P_{k+1} is a polynomial of degree $k+1$. One can represent P_{k+1} as a linear combination of $P_0, P_1, \ldots, P_k, P_{k+1}$, and correspondingly, $\mathcal{F}[h_{k+1}] = P_{k+1} h_0$ is represented as a linear combination

$$\mathcal{F}[h_{k+1}] = \sum_{j=0}^{k+1} b_j h_j \quad \text{with} \quad b_j = const.$$

However, by virtue of (ii), (iv), and our assumption $\mathcal{F}[h_j] = c_j h_j$ for $j \leq k$, we have

$$b_j \int h_j^2 dx = \int h_j \cdot \mathcal{F}[h_{k+1}] \, dx = \int h_{k+1} \cdot \mathcal{F}[h_j] \, dx = c_j \int h_{k+1} h_j \, dx = 0 \quad \text{for} \quad j = 0, 1, \ldots, k.$$

Hence $b_0 = b_1 = \ldots = b_k = 0$, and we obtain the desired equality $\mathcal{F}[h_{k+1j}] = c_{k+1} h_{k+1}$ with $c_{k+1} = b_{k+1}$.□

Alternative Proof. Differentiating the equality $h_k(x) = e^{\frac{x^2}{2}} (-D_x)^k (e^{-x^2})$, we get

$$D_x h_k = x h_k - h_{k+1}, \quad h_{k+1} = (x - D_x) h_k.$$

Assuming $\mathcal{F}[h_k] = c_k h_k$ and using properties (iii), we extend this equality for $k+1$:

$$\mathcal{F}[h_{k+1}] = \mathcal{F}[(x - D_x) h_k] = i(D_\xi - \xi) \mathcal{F}[h_k] = i c_k (D_\xi - \xi) h_k = -i c_k h_{k+1} = c_{k+1} h_{k+1},$$

where $c_{k+1} = -i c_k$. From this recurrent equality together with $c_0 = \sqrt{2\pi}$ it follows

$$c_k = (-i)^k \sqrt{2\pi} \quad \text{for all} \quad k = 0, 1, 2, \ldots.$$

□