Math 8583: Theory of Partial Differential Equations: Fall 2003
Solutions for Midterm Exam

1. Let $u_1, u_2, \ldots$ be sequence of harmonic functions, which is defined on the ball $B_2 := \{x \in \mathbb{R}^n : \ |x| < 2\}$ and converges to a harmonic function u in $L^2(B_2)$, i.e.

$$\int\limits_{B_2} |u_k - u|^2 dx \to 0 \quad \text{as} \quad k \to \infty.$$

Show that $\sup\limits_{|x| \le 1} |u_k - u| \to 0$ as $k \to \infty$.

Solution is similar to Drill Problem 1 (iv). $\square$

2. Show that there are no positive classical solutions $u = u(t, x)$ to the Cauchy problem

$$u_t = u_{xx} \quad \text{in} \quad (0,1) \times \mathbb{R}^1, \quad u(0, x) \equiv \exp(x^4).$$

Solution: see solution to Drill Problem 2.

3. Let $f(x) = e^{-|x|^2}$ on $\mathbb{R}^n$. Find the convolution $f * f$.

Solution. We know that the fundamental solution to the heat equation,

$$\Gamma\,(t,x) = (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{x^2}{4t}}, \quad t > 0, \quad \text{has Fourier transform} \quad \mathcal{F}[\Gamma\,(t,\cdot)](\xi) = e^{-t\xi^2}.$$

Since $f(x) = e^{-|x|^2} = \pi^{\frac{n}{2}}\Gamma\left(\frac{1}{4}, x\right)$, we have

$$\begin{aligned} \mathcal{F}[f * f] &= \mathcal{F}[f] \cdot \mathcal{F}[f] = \left(\pi^{\frac{n}{2}} e^{-\frac{\xi^2}{4}}\right)^2 = \pi^n e^{-\frac{\xi^2}{2}} = \mathcal{F}\left[\pi^n \Gamma\left(\frac{1}{2}, \cdot\right)\right], \\ f * f(x) &= \pi^n \Gamma\left(\frac{1}{2}, x\right) = \left(\frac{\pi}{2}\right)^{n/2} e^{-\frac{x^2}{2}}. \end{aligned}$$

$\square$

4. Let u, v be functions with compact support in $(-1, 1) \in \mathbb{R}^1$, such that

$$[u]_{0,\alpha} = \sup_{x \ne y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty, \quad \text{and} \quad [v]_{0,\beta} < \infty,$$

with some constants $\alpha, \beta \in (0,1)$. Show that in the case $\alpha + \beta < 1$, the function $w := u * v$ satisfies

$$[w]_{0,\alpha+\beta} < \infty.$$

Solution: see solution to Drill Problem 5 (i).

5. For given constants $A > 0$ and $R > 0$, let Ω be the cylinder

$$\Omega := \{x = (x', x_n) \in \mathbb{R}^n : |x'| < 1,\ |x_n| < A\}.$$

Let u be a function in $C^2(\overline{\Omega})$, such that

$$u \ge 0, \quad Lu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u \le 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad u \ge 1 \quad \text{on} \quad (\partial\Omega) \cap \{|x_n| = A\},$$

where the matrix $a = [a_{ij}]$ satisfies the uniform ellipticity condition

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \le (a\xi, \xi) \le \nu^{-1}|\xi|^2,$$

with a constant $\nu \in (0, 1]$. Show that

$$u(0) \ge c = c(n, \nu, A) > 0.$$

Solution. We will solve this problem in two different ways.
1st solution. Introduce the function

$$v(x) = v(x', x_n) := q^{\mu} \cosh(\lambda x_n), \quad \text{where} \quad q := 1 - |x'|^2 = 1 - \sum_{j=1}^{n-1} x_j^2,$$

and the constants $\mu \ge 2$ and $\lambda \gg 1$ will be selected below. For $1 \le i, j \le n-1$, we have

$$\begin{aligned} D_i v &= -2\mu x_i \cdot q^{\mu-1} \cosh(\lambda x_n), \quad D_{ij} v = [4\mu(\mu-1) x_i x_j - 2\mu q \delta_{ij}] \cdot q^{\mu-2} \cosh(\lambda x_n), \\ D_{in} v &= -2\mu\lambda x_i \cdot q^{\mu-1} \sinh(\lambda x_n), \quad D_{nn} v = \lambda^2 q^{\mu} \cosh(\lambda x_n). \end{aligned}$$

Further, denote $w := q^{\mu-2} \cosh(\lambda x_n)$ and

$$\xi_i := -2\theta\mu x_i \quad \text{for} \quad i \le n-1, \quad x_n := \lambda q, \quad \text{where} \quad \theta := \tanh(\lambda x_n) \in (-1, 1).$$

Then

$$\begin{aligned} D_{ij} v &= [\xi_i \xi_j + 4\mu^2(1-\theta^2) x_i x_j - 4\mu x_i x_j - 2\mu q \delta_{ij}] \cdot w \quad \text{for} \quad i, j \le n-1, \\ D_{in} v &= \xi_i \xi_n \cdot w \quad \text{for} \quad i \le n-1, \quad D_{nn} v = \xi_n^2 \cdot w. \end{aligned}$$

Using matrix notations with matrices $a := [a_{ij}]_{i,j=1}^{n}$ and $a' := [a_{ij}]_{i,j=1}^{n-1}$, we can write

$$Lv = (aDv, Dv) = [(a\xi, \xi) + 4(1-\theta^2)\mu^2 \cdot (a'x', x') - 4\mu \cdot (a'x', x') - 2\mu q \cdot tr\, a'] \cdot w.$$

Here

$$\begin{aligned} (a\xi, \xi) &\ge \nu|\xi|^2 = 4\nu\theta^2\mu^2 \cdot |x'|^2 + \nu\lambda^2 q^2, \\ \nu \cdot |x'|^2 &\le (a'x', x') \le \nu^{-1} \cdot |x'|^2, \quad tr\, a' \le (n-1)\nu^{-1}. \end{aligned}$$

Then

$$Lv \geq [\nu\lambda^2 q^2 + 4\nu\mu^2 \cdot |x'|^2 - 4\nu^{-1}\mu \cdot |x'|^2 - 2(n-1)\nu^{-1}\mu q] \cdot w = Qw,$$

where

$$Q = Q(\mu, \lambda, q) := \nu\lambda^2 q^2 + 4\mu \cdot (\nu\mu - \nu^{-1}) \cdot (1-q) - 2(n-1)\nu^{-1}\mu q.$$

It is easy to see that one can choose $\mu = \mu(n, \nu) \gg 1$, such that

$$Q(\mu, \lambda, q) \geq Q(\mu, 0, q) \geq 0 \quad \text{for} \quad 0 \leq q \leq \frac{1}{2} \quad \text{and all } \lambda,$$

and then $\lambda = \lambda(n, \nu) \gg 1$, such that

$$Q(\mu, \lambda, q) \geq 0 \quad \text{for} \quad \frac{1}{2} \leq q \leq 1.$$

Then $Lv \geq Qw \geq 0 \geq Lu$ in $\Omega = \{|x'| < 1, \; |x_n| < A\}$, and also

$$u \geq cv \quad \text{on} \quad \partial\Omega, \quad \text{where} \quad c = c(n, \nu, A) := \frac{1}{\cosh(\lambda A)} > 0.$$

By the comparison principle, $u \geq cv$ in Ω, hence $u(0) \geq cv(0) = c > 0$. $\square$

2nd solution involves more general arguments, which also work in many other situations. After we prove the following lemma and its corollary, the solution itself becomes very short.

Lemma. *There exists a constant* $\gamma = \gamma(n, \nu) > 0$ *such that the function* $v(x) := |x|^{-\gamma}$ *satisfies* $Lv(x) \geq 0$ *for* $x \neq 0$, *where* $Lv := (aD, Dv)$ *and* $a = [a_{ij}(x)]$ *is the matrix function satisfying the uniform ellipticity condition*

$$\nu|\xi|^2 \leq (a\xi, \xi) \leq \nu^{-1}|\xi|^2 \quad \textit{for all} \quad \xi \in \mathbb{R}^n,$$

with a constant $\nu \in (0, 1]$.

Proof. We have

$$\begin{aligned} D_i v(x) &= -\gamma|x|^{-\gamma-2} x_i, \quad D_{ij} v(x) = \gamma|x|^{-\gamma-2} \cdot \left[(\gamma+2)\frac{x_i x_j}{|x|^2} - \delta_{ij}\right], \\ Lv(x) &= \gamma|x|^{-\gamma-2} \cdot \left[(\gamma+2)\frac{(ax, x)}{|x|^2} - tr\, a\right] \geq \gamma|x|^{-\gamma-2} \cdot \left[(\gamma+2)\nu - n\nu^{-1}\right] > 0 \end{aligned}$$

for $x \neq 0$, provided $\gamma \geq n\nu^{-2} > 0$. $\square$

Corollary. *Let* $u \in C^2_{loc}(\Omega) \cap C(\bar{\Omega})$, $u \geq 0$ *and* $Lu = (aD, Du) \leq 0$ *in* Ω. *Let* $y \in R^n$ *and* $r > 0$ *be such that*

$$u \geq M = const > 0 \quad \text{on} \quad (\partial\Omega) \cap B_{4r}(y), \quad \text{and} \quad \Omega \cap B_r(y) \cap \{u < M\} = \emptyset.$$

Then

$$\Omega \cap B_{2r}(y) \cap \{u < \theta M\} = \emptyset,$$

where the constant $\theta = \theta(n, \nu) \in (0, 1)$. *In other words,* $u \geq \theta M$ *on the set* $\Omega \cap B_{2r}(y)$, *if this set is non-empty.*

Proof. Without loss of generality, we may assume $y = 0, r = 1$, and $M = 1$. Consider the function

$$w(x) := \frac{|x|^{-\gamma} - 4^{-\gamma}}{1 - 4^{-\gamma}} \quad \text{on the set} \quad \Omega' := \Omega \cap B_4(0) \cap \{u < 1\},$$

where $\gamma = \gamma(n, \nu) > 0$ is the constant in the previous lemma. We assume that Ω' is non-empty, because otherwise there is nothing to prove. By our assumptions,

$$\Omega' \cap B_1(0) = \Omega \cap B_1(0) \cap \{u < 1\} = \emptyset,$$

hence $\partial\Omega' \subset \overline{B_4(0)} \setminus B_1(0)$. Moreover,

$$\begin{array}{rcll} u & = & 1 \geq w & \text{on} \quad (\partial\Omega') \cap (B_4(0) \setminus B_1(0)), \\ u & \geq & 0 = w & \text{on} \quad (\partial\Omega') \cap \partial B_4(0), \end{array}$$

i.e. $u \geq w$ on $\partial\Omega'$. In addition, $Lu \leq 0 \leq Lw$ in Ω'. By the comparison principle, we get $u \geq w$ in Ω'. Note that

$$w(x) \geq \theta = \theta(n, \nu) := \frac{2^{-\gamma} - 4^{-\gamma}}{1 - 4^{-\gamma}} \in (0, 1) \quad \text{for} \quad 0 < |x| \leq 2,$$

hence $u \geq w \geq \theta$ in $\Omega' \cap B_2(0)$, which yields the desired property

$$\Omega \cap B_2(0) \cap \{u < \theta\} = \Omega' \cap B_2(0) \cap \{u < \theta\} = \emptyset.$$

Corollary is proved. □

Solution to Problem 5. Fix the minimal natural number $m \geq 4A$, take $r := \frac{A}{m} \in \left(0, \frac{1}{4}\right]$, and consider the sequence of balls

$$B_r(P_j), \quad j = 0, 1, \cdots, m, \quad \text{where} \quad P_j := (0, (m + 1 - j)r) \in \mathbb{R}^n.$$

The union of these balls looks like a "caterpillar" with its "head" $B_r(P_0)$ being outside of $\Omega = \{|x'| < 1, |x_n| < A = mr\}$, the "neck" $B_r(P_1)$ penetrating through the disk $\{|x'| < 1, x_n = A\}$ which is a portion of $\partial\Omega$ with boundary values $u \geq 1$, the rest of the "body", $B_r(P_j) \subset \Omega$ for $j = 2, 3, \cdots, m$, and finally, the "tail" $B_r(P_m)$ touching the origin, i.e. $0 \in \partial B_r(P_m)$. Using the previous corollary with $y = P_0$, $M = 1$, we get

$$u \geq \theta = \theta(n, \nu) \in (0, 1) \quad \text{on} \quad \Omega \cap B_{2r}(P_0) \supset \Omega \cap B_r(P_1).$$

Then we can use it again with $y = P_1$, $M = \theta$:

$$u \geq \theta^2 \quad \text{on} \quad \Omega \cap B_{2r}(P_1) \supset B_r(P_2).$$

Continuing this procedure, we have $u \geq \theta^j$ on $B_r(P_j)$ for $j = 2, 3, \cdots, m$, and finally,

$$u(0) \geq \inf_{B_r(P_m)} u \geq \theta^m =: c = c(n, \nu, A) > 0.$$

□