Math 8583: Theory of Partial Differential Equation: Fall 2003

Homework 2. Problems and Solutions

1. Show that $(n-1)$- dimensional surface area of $\partial B_1 := \{x \in \mathbb{R}^n : |x| = 1\}$ is

$$\omega_n := |\partial B_1| = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}, \quad \text{where} \quad \Gamma(a) := \int_0^\infty t^{a-1} e^{-t} dt \quad \text{for} \quad a > 0.$$

Solution. Using the polar coordinates in $\mathbb{R}^n$, and then substituting $r = (2t)^{\frac{1}{2}}$, we obtain

$$\begin{aligned}(2\pi)^{\frac{n}{2}} &= \int_{\mathbb{R}^n} \exp\left(-\frac{x^2}{2}\right) dx = \int_0^\infty dr \int_{\partial \mathbb{B}_r} \exp\left(-\frac{r^2}{2}\right) dS_x = \int_0^\infty |\partial B_r| \cdot \exp\left(-\frac{r^2}{2}\right) dr \\ &= |\partial B_1| \cdot \int_0^\infty r^{n-1} \cdot \exp\left(-\frac{r^2}{2}\right) dr = |\partial B_1| \cdot 2^{\frac{n}{2}-1} \int_0^\infty t^{\frac{n}{2}-1} e^{-t} dt = |\partial B_1| \cdot 2^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right),\end{aligned}$$

and the desired equality follows. □

2. Let $u(x)$ be a harmonic function in a ball $B_R := \{x \in \mathbb{R}^n : |x| < R\}$. Show that for $0 < a \le b \le c < R$, with $b^2 = ac$, we have

$$\int_{\partial B_1} u(a\omega) \cdot u(c\omega)\, dS_\omega = \int_{\partial B_1} u^2(b\omega)\, dS_\omega.$$

Solution. We know (see solutions to Drill Problem 1 (i–iv)) that the function u is represented in the form

$$u(x) = \sum_{k=0}^\infty P_k(x) \quad \text{in} \quad B_R \tag{1}$$

where for each k, P_k is a *homogeneous harmonic polynomial* of degree k. Moreover, the convergence in (1) is uniform on any ball B_r with $r < R$, and polynomials $\{P_k\}$ are orthogonal in $L^2(\partial B_r)$:

$$\int_{\partial B_r} P_j P_k\, dS_x = 0 \quad \text{for} \quad j \ne k, \quad r > 0.$$

The uniform convergence of the series (1) means

$$\lim_{m\to\infty} \sup_{B_r} |S_m - u| = 0 \quad \text{for} \quad 0 < r < R, \quad \text{where} \quad S_m = S_m(x) := \sum_{k=0}^m P_k(x).$$

For any $m \geq 0$, we have

$$\begin{aligned} L_m \ &:= \int_{\partial B_1} S_m(a\omega)\cdot S_m(c\omega)\,dS_\omega = \int_{\partial B_1} \sum_{j=0}^m P_j(a\omega)\cdot \sum_{k=0}^m P_k(c\omega)\,dS_\omega \\ &= \sum_{j,k=0}^m a^j c^k \int_{\partial B_1} P_j(\omega)P_k(\omega)\,dS_\omega = \sum_{k=0}^\infty (ac)^k \int_{\partial B_1} P_k^2(\omega)\,dS_\omega = \sum_{k=0}^\infty b^{2k} \int_{\partial B_1} P_k^2(\omega)\,dS_\omega, \\ R_m \ &:= \int_{\partial B_1} S_m^2(b\omega)\,dS_\omega = \int_{\partial B_1} \sum_{j=0}^m P_j(b\omega)\cdot \sum_{k=0}^m P_k(b\omega)\,dS_\omega \\ &= \sum_{j,k=0}^m b^{j+k} \int_{\partial B_1} P_j(\omega)P_k(\omega)\,dS_\omega = \sum_{k=0}^\infty b^{2k} \int_{\partial B_1} P_k^2(\omega)\,dS_\omega, \end{aligned}$$

i.e. $L_m = R_m$ for all m. By the uniform convergence $S_m \to u$ on compact subsets of B_R, L_m and R_m converge correspondingly to the left and right sides in our formula:

$$\int_{\partial B_1} u(a\omega)\cdot u(c\omega)\,dS_\omega = \lim_{m\to\infty} L_m = \lim_{m\to\infty} R_m = \int_{\partial B_1} u^2(b\omega)\,dS_\omega.$$

□

3. Let $f(x)$ be a function in $C_0^\infty(\mathbb{R}^1)$. For $0 < \alpha \leq 1$, define

$$[f]_\alpha := \sup_{x,h} \frac{|f(x+h)-f(x)|}{|h|^\alpha}, \quad [f]_\alpha^* := \sup_{x,h} \frac{|f(x+h)-2f(x)+f(x-h)|}{|h|^\alpha}. \tag{2}$$

(i) Show that $[f]_\alpha^* \leq 2\cdot [f]_\alpha$.
(ii) For $0 < \alpha < 1$, show that $[f]_\alpha \leq N \cdot [f]_\alpha^*$, with a constant N depending only on α.

Solution. For functions f defined on $\mathbb{R}^1$, and $x, h \in \mathbb{R}^1$, define $T_h f(x) := f(x+h)$, $If(x) := T_0 f(x) \equiv f(x)$. It is easy to see that

$$[f]_\alpha = \sup_{h>0} h^{-\alpha} ||(T_h - I)f||_\infty, \qquad [f]_\alpha^* = \sup_{h>0} h^{-\alpha} ||(T_h-I)^2 f||_\infty, \quad \text{where} \quad ||\cdot||_\infty := \sup_{\mathbb{R}^1} |\cdot|.$$

(i) follows immediately from

$$||(T_h-I)^2 f||_\infty = ||(T_h-I)((T_h-I)f)||_\infty \leq 2||(T_h-I)f||_\infty.$$

(ii) Using the algebraic identities

$$T_h - I = \frac{1}{2}[(T_h^2 - I) - (T_h - I)^2], \quad T_h^2 = T_{2h},$$

we can write

$$||(T_h-I)f||_\infty = \frac{1}{2}||(T_{2h}-I)f - (T_h-I)^2 f||_\infty \leq \frac{1}{2}||(T_{2h}-I)f||_\infty + \frac{1}{2}||(T_h-I)^2 f||_\infty,$$

and for arbitrary $h > 0$,

$$\begin{aligned} h^{-\alpha}||(T_h - I)f||_\infty &\le 2^{\alpha-1} \cdot (2h)^{-\alpha}||(T_{2h} - I)f||_\infty + 2^{-1} \cdot h^{-\alpha}||(T_h - I)^2 f||_\infty \\ &\le 2^{\alpha-1} \cdot [f]_\alpha + 2^{-1} \cdot [f]^*_\alpha. \end{aligned}$$

Note that the right-hand side does not depend on $h > 0$. Since $f \in C_0^\infty$, the seminorm $[f]_\alpha$ is finite, and moreover,

$$[f]_\alpha = \sup_{h>0} h^{-\alpha}||(T_h - I)f||_\infty \le 2^{\alpha-1} \cdot [f]_\alpha + 2^{-1} \cdot [f]^*_\alpha.$$

Finally, since $\alpha < 1$, we have $2^{\alpha-1} < 1$, and

$$[f]_\alpha \le N \cdot [f]^*_\alpha, \quad \text{where} \quad N := \frac{2^{-1}}{1 - 2^{\alpha-1}} = \frac{1}{2 - 2^\alpha} < \infty.$$

□

4. Consider the function

$$f(x) := x \ln|x| \quad \text{for} \quad 0 < |x| < 1, \quad f(x) \equiv 0 \quad \text{otherwise}.$$

Show that the quantities in (2) with $\alpha = 1$ are such that $[f]_1 = \infty$, while $[f]^*_1 < \infty$.

Solution. The first property of the function f is obvious:

$$[f]_1 := \sup_{x,h} \frac{|f(x+h) - f(x)|}{|h|} \ge \sup_{0<h<1} \frac{|f(h) - f(0)|}{h} = \sup_{0<h<1} |\ln h| = \infty.$$

In order to prove the estimate $[f]^*_1 \le N = const < \infty$, we need to demonstrate that the function

$$F(x,h) := f(x+h) - 2f(x) + f(x-h) \quad \text{satisfies} \quad |F(x,h)| \le Nh \tag{*}$$

for all $x \in \mathbb{R}^1$ and $h > 0$. Since f is bounded, it suffices to consider $0 < h < \frac{1}{4}$. Moreover, by symmetry we can assume $x \ge 0$. If $x \ge \frac{1}{2}$, then

$$|F(x,h)| \le |f(x+h) - f(x)| + |f(x-h) - f(x)| \le 2Kh,$$

where $K := \sup_{(1/4,1)} |f'| > 0$ is the Lipschitz constant of the function f on $[\frac{1}{4}, \infty)$. Now it remains to prove the estimate (*) for $0 \le x < \frac{1}{2},\ 0 < h < \frac{1}{4}$. We will do it in two different ways: **(a)** by direct calculation, and **(b)** using the mean value theorem.

(a) Direct calculation. In the interval $[-1,1]$, the function $f(x)$ is continuous and has derivative $f'(x) = \ln|x| + 1$ for $0 < |x| < 1$. Consequently, for fixed $0 < h < \frac{1}{4}$, the functions $F(x,h)$ is continuous in the interval $(-\frac{3}{4}, \frac{3}{4})$, and its derivative

$$\begin{aligned} F'_x(x,h) &= f'(x+h) - 2f'(x) + f'(x-h) = \ln|x+h| - 2\ln|x| + \ln|x-h| \\ &= \ln\left|\frac{x^2 - h^2}{x^2}\right| = \ln\left|1 - \frac{h^2}{x^2}\right| \quad \text{for} \quad x \in \left(-\frac{3}{4}, \frac{3}{4}\right) \setminus \{-h, 0, h\}. \end{aligned}$$

We see that

$$F'_x(x,h) > 0 \quad \text{for} \quad 0 < |x| < ch, \quad F'_x(x,h) < 0 \quad \text{for} \quad ch < |x| < \frac{3}{4}, \quad \text{where} \quad c := 2^{-\frac{1}{2}}.$$

therefore,

$$\begin{aligned}
\sup_{-\frac{3}{4}<x<\frac{3}{4}} \frac{F(x,h)}{h} &= \frac{F(ch,h)}{h} = (c+1)\ln((c+1)h) - 2c\cdot\ln(ch) + (c-1)\ln((1-c)h) \\
&= (c+1)\ln(c+1) - 2c\cdot\ln c + (c-1)\ln(1-c) \\
&= c\cdot\ln\left(\frac{1-c^2}{c^2}\right) + \ln\left(\frac{1+c}{1-c}\right) = \ln\left(\frac{1+c}{1-c}\right) \\
&= \ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) = \ln(3+2\sqrt{2}).
\end{aligned}$$

Since $F(-x,h) \equiv -F(x,h)$, we obtain the desired estimate

$$|F(x,h)| \le Nh \quad \text{for} \quad 0 \le x < \frac{1}{2},\ 0 < h < \frac{1}{4},$$

with $N := \ln(3+2\sqrt{2})$.

(b) Proof by the mean value theorem. Our argument will be different in the cases $x \ge 2h$ and $x < 2h$.

(b1) $0 < 2h \le x < \frac{1}{2}$. Note that

$$f(x) = x\ln x, \quad f'(x) = 1 + \ln x, \quad f''(x) = \frac{1}{x} \quad \text{for} \quad 0 < x < 1.$$

Applying the mean value theorem twice, we get

$$F(x,h) = g(x+h) - g(x) = h\cdot g'(x_1) = h\cdot[f'(x_1) - f'(x_1-h)] = h^2\cdot f''(x_2),$$

where $x_1 \in (x, x+h),\ x_2 \subset (x_1 - h, x_1)$, hence

$$0 < f''(x_2) = \frac{1}{x_2} < \frac{1}{h}, \quad \text{and} \quad 0 < F(x,h) = h^2\cdot f''(x_2) < h^2\cdot\frac{1}{h} = h.$$

(b2) $0 \le x < 2h < \frac{1}{2}$. In this case, the points $x+h, x, x-h \in (-3h, 3h) \subset (-\frac{3}{4}, \frac{3}{4})$. Note that $F(x,h)$ does not change if we replace f by $\tilde{f} = f - \lambda$ with an arbitrary linear function $\lambda = \lambda(x) = c_1x + c_2$. Take $\lambda(x) = x\cdot\ln(3h)$. Then

$$|F(x,h)| = |\tilde{f}(x+h) - 2\tilde{f}(x) + \tilde{f}(x-h)| \le 4\cdot \sup_{(-3h,3h)} |\tilde{f}| = 4\cdot\sup_{(0,3h)}|f-\lambda|.$$

By the mean value theorem, for arbitrary $y \in (0, 3h)$, we have

$$\lambda(y) - f(y) = y\cdot[\ln(3h) - \ln y] = y\cdot(3h-y)\cdot\frac{1}{y_1}, \quad \text{where} \quad y_1 \in (y, 3h),$$

and $0 < \lambda(y) - f(y) < 3h - y < 3h$. Therefore, $|F(x,h)| \le 4\cdot 3h = 12h$.

We have proved the estimate $|F(x,h)| \le Nh$ for arbitrary $x \in \mathbb{R}^1$ and $h > 0$, i.e. $[f]_1^* \le N < \infty$. $\square$

5. For $r > 0$, denote $\Omega_r := \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_2| < x_1 < r\}$. Let u be a function in $C^2(\overline{\Omega_R})$, $R > 0$, such that

$$Lu = \sum_{i,j=1}^{2} a_{ij} D_{ij} u = 0 \quad \text{in} \quad \Omega_R, \quad \text{and} \quad u = 0 \quad \text{on} \quad (\partial\Omega_R) \cap \{|x_2| = x_1\},$$

where the matrix $a = [a_{ij}(x)]$ satisfies the uniform ellipticity condition

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \le (a\xi, \xi) \le \nu^{-1}|\xi|^2,$$

with a constant $\nu \in (0, 1]$. Show that

$$M_r := \sup_{\Omega_r} |u| \le \left(\frac{2r}{R}\right)^{1+\alpha} M_R \quad \text{for} \quad 0 < r \le R, \tag{**}$$

where the constant $\alpha > 0$ depends only on ν.

Solution. This problem is easily reduced to the estimate

$$M_r \le 2^{-1-\alpha} M_{2r} \quad \text{for} \quad 0 < 2r \le R. \tag{***}$$

Indeed, for arbitrary $r \in (0, R]$, there exists an integer $k \ge 0$ such that $\frac{R}{2} < 2^k r \le R$. If $k = 0$, then $R < 2r$ and (**) is obvious. If $k \ge 1$, then by iterating (***) we get

$$M_r \le 2^{-(1+\alpha)} M_{2r} \le 2^{-2(1+\alpha)} M_{2^2 r} \le \cdots \le 2^{-k(1+\alpha)} M_{2^k r} \le 2^{-k(1+\alpha)} M_R.$$

The inequality $\frac{R}{2} < 2^k r$ implies $2^{-k} \le \frac{2r}{R}$, $2^{-k(1+\alpha)} \le \left(\frac{2r}{R}\right)^{1+\alpha}$, and the estimate (**) follows.

This it suffices to prove the estimate (***) with a constant $\alpha = \alpha(\nu) > 0$. For the proof of this estimate, we can assume $r = 2$, because the general case follows by rescaling $x \to const \cdot x$. Consider the functions

$$U(x) = U(x_1, x_2) := \frac{1}{4} M_4 x_1 \pm u(x_1, x_2) \quad \text{in} \quad \Omega_4 := \{\, |x_2| < x_1 < 4\}.$$

It is easy to see that $LU = 0$ in Ω_4 and $U \ge 0$ on $\partial\Omega_4$. Then by the maximum (or comparison) principle, we get $U \ge 0$ in Ω_4, in particular,

$$\pm u \le \frac{1}{4} M_4 x_1 \le \frac{1}{2} M_4 \quad \text{in} \quad \Omega_2 := \{\, |x_2| < x_1 < 2\},$$

i.e. $M_2 \le 2^{-1} M_4$. The desired estimate (***) is a bit stronger: we heed to replace 2^{-1} with a smaller constant $2^{-1-\alpha}$, $\alpha > 0$. For this purpose, one can use either of the two methods of solving Problem 5 in Midterm Exam.

Here we use the first of these two methods, which is based on the properties of the function

$$v(x_1, x_2) := (1 - x_1^2)^\mu \cosh(\lambda x_2).$$

We know (see solutions for Midterm Exam) that one can choose the constants $\mu = \mu(\nu) > 0$ and $\lambda = \lambda(\nu) > 0$ in such a manner that

$$Lv = \sum_{i,j=1}^{2} a_{ij} D_{ij} v \ge 0 \quad \text{in} \quad (-1, 1) \times \mathbb{R}^1$$

for any matrix $a = [a_{ij}(x)]$ satisfying the uniform ellipticity condition with the constant $\nu \in (0, 1]$. Then obviously

$$Lv(x_1 - 2, x_2) \geq 0 \quad \text{in} \quad \Omega' := \{1 < x_1 < 3,\ |x_2| < x_1\} \subset \Omega_4.$$

Since

$$\begin{aligned} U(x_1, x_2) &\geq \frac{1}{4} M_4 \geq cM_4 \cdot v(x_1 - 2, x_2) \quad \text{for} \quad 1 < x_1 < 3,\ |x_2| = x_1, \\ U(x_1, x_2) &\geq 0 = cM_4 \cdot v(x_1 - 2, x_2) \quad \text{for} \quad x_1 = 1 \text{ or } 3,\ |x_2| < x_1, \end{aligned}$$

where $c = c(\nu) := \frac{1}{4\cosh(3\lambda)} > 0$, we have

$$U(x_1, x_2) \geq cM_4 \cdot v(x_1 - 2, x_2) \quad \text{on} \quad \partial\Omega', \quad \text{and} \quad LU(x_1, x_2) = 0 \leq cM_4 \cdot Lv(x_1 - 2, x_2) \quad \text{in} \quad \Omega'.$$

By the comparison principle, $U(x_1, x_2) \geq cM_4 \cdot v(x_1 - 2, x_2)$ in Ω', in particular,

$$2^{-1} M_4 \pm u(x_1, x_2) = U(2, x_2) \geq cM_4 \quad \text{for} \quad |x_2| \leq 2.$$

Choosing the constant $\alpha = \alpha(\nu) > 0$ from the relation $2^{-1} - c = 2^{-1-\alpha}$, we obtain $2^{-1-\alpha} M_4 \pm u(x_1, x_2) \geq 0$ on $\partial\Omega_2$, and by the comparison principle,

$$M_2 := \sup_{\Omega_2} |u| \leq 2^{-1-\alpha} M_4,$$

which is the estimate (***). □