Math 8583: Theory of Partial Differential Equations: Fall 2007
Midterm Exam, November 7. Problems and Solutions

1. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be a functions satisfying

$$v > 0, \quad \Delta v + \lambda v = 0 \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega,$$

where λ =const. Show that the problem

$$\Delta u + \lambda u = 1 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

cannot have solutions in $C^2(\Omega) \cap C(\overline{\Omega})$.

Solution. If $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $\partial\Omega \in C^1$, one can get a desired contradiction using Green's formula:

$$0 < \int\limits_\Omega v\,dx = \int\limits_\Omega v(\Delta u + \lambda u)\,dx = \int\limits_\Omega u(\Delta v + \lambda v)\,dx = 0.$$

In general, Ω is an arbitrary bounded domain in $\mathbb{R}^n$, and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$. Note that for any $\varepsilon > 0$, the function $(u(x) - \varepsilon)/v(x) \to -\infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$. Therefore, this function attains its maximum on Ω at some interior point $x_0 \in \Omega$:

$$M := \sup_\Omega \frac{u - \varepsilon}{v} = \frac{u - \varepsilon}{v}(x_0).$$

Then the function

$$u_\varepsilon := u - \varepsilon - Mv \le 0 \quad \text{in} \quad \Omega, \qquad \text{and} \quad u_\varepsilon(x_0) = 0.$$

At the point of maximum, we must have

$$0 \ge \Delta u_\varepsilon(x_0) = (\Delta u - M\Delta v)(x_0) = (-\lambda u + 1 + M\lambda v)(x_0) = -\lambda u_\varepsilon(x_0) + 1 - \lambda\varepsilon = 1 - \lambda\varepsilon > 0$$

if $\varepsilon > 0$ is small enough. This contradiction proves our statement.

2. For a fixed constant $a \in (0, 1)$, find

$$\inf_{\mathcal{A}} \int\limits_a^1 r\left[f'(r)\right]^2 dr, \quad \text{where} \quad \mathcal{A} := \Big\{ f \in C^1([a, 1]),\ f(a) = 1,\ f(1) = 0 \Big\}.$$

Solution. Using polar coordinates, we can write

$$\int\limits_a^1 r\left[f'(r)\right]^2 dr = \frac{1}{2\pi} \int\limits_{\{a<|x|<1\}} |Du(x)|^2 dx, \quad \text{where} \quad u(x) := f(|x|).$$

It is known that the right side attains its minimum on the solution of the problem

$$\Delta u = 0 \quad \text{in} \quad B_1 \setminus \overline{B_a}, \qquad u = 1 \quad \text{on} \quad \partial B_a, \quad u = 0 \quad \text{on} \quad \partial B_1.$$

The solution of this problem is $u(x) := \ln|x|/\ln a$, which corresponds to $f(r) := \ln r/\ln a$. Then the minimum of the given functional is

$$\int_a^1 r\big[f'(r)\big]^2 dr = \int_a^1 r\left(\frac{1}{r\ln a}\right)^2 dx = \frac{-1}{\ln a}.$$

3. Describe all the vectors $\boldsymbol{v} = (\alpha, \beta) \in \mathbb{R}^2$ such that the function

$$u(x_1, x_2, x_3, x_4) := \Big[x_1^2 + x_2^2 + (\alpha x_3 + \beta x_4)^2\Big]^{-1/2}$$

is harmonic for $x_1^2 + x_2^2 > 0$.

Solution. Note that the function $w(y_1, y_2, y_3) := |y|^{-1}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$. The given function u can be written as

$$u(x_1, x_2, x_3, x_4) = w(x_1, x_2, \alpha x_3 + \beta x_4).$$

By the chain rule

$$\begin{aligned} \Delta u &:= D_{11}u + D_{22}u + D_{33}u + D_{44}u = D_{11}w + D_{22}w + (\alpha^2 + \beta^2)D_{33}w \\ &= \Delta w + (\alpha^2 + \beta^2 - 1)D_{33}w = (\alpha^2 + \beta^2 - 1)D_{33}w. \end{aligned}$$

Hence u is harmonic for $x_1^2 + x_2^2 > 0$ if and only if $\alpha^2 + \beta^2 = 1$, i.e. $|\boldsymbol{v}| = 1$.

4 (a). Let $\alpha \in (0, 1)$ be a fixed constant. Show that there is a constant $c = c(n, \alpha) > 0$ such that the function

$$v(x) := |x|^\alpha + c \cdot x_n^\alpha \quad \text{satisfies} \quad \Delta v \le 0 \quad \text{in} \quad \mathbb{R}^n_+ := \{x = (x_1, \dots, x_{n-1}, x_n) : \ x_n > 0\}.$$

4 (b). Let Ω be a bounded open convex set in $\mathbb{R}^n$, and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical solution to the problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad u = g \quad \text{on} \quad \partial\Omega,$$

where g satisfies

$$|g(x) - g(y)| \le |x - y|^\alpha \quad \text{for all} \quad x, y \in \mathbb{R}^n, \quad \text{with} \quad \alpha = \text{const} \in (0, 1).$$

Show that there is a constant $K > 0$ such that

$$|u(x) - u(y)| \le K \cdot |x - y|^\alpha \quad \text{for all} \quad x \in \Omega, \ y \in \partial\Omega.$$

4 (c). Show that the previous estimate holds true for all $x, y \in \Omega$.

Solution. In the case $n = 1$, all the statements are true with $c = 0$ and $K = 1$. Therefore, we only consider the case $n \ge 2$.

(a). Denote $r := |x|$. Then

$$\begin{aligned} \Delta(|x|^\alpha) &= \Delta(r^\alpha = r^{1-n}\big(r^{n-1}(r^\alpha)'\big)' = \alpha(\alpha + n - 2)\, r^{\alpha-2}, \\ \Delta\big(x_n^\alpha\big) &= D_{nn}\big(x_n^\alpha\big) = \alpha(\alpha - 1)\, x_n^{\alpha-2}. \end{aligned}$$

Taking $c := (\alpha + n - 2)/(1-\alpha) > 0$, we get

$$\Delta\big(|x|^\alpha + c x_n^\alpha\big) = \alpha(\alpha+n-2)\big(r^{\alpha-2} - x_n^{\alpha-2}\big) \le 0 \quad \text{in} \quad \mathbb{R}^n_+.$$

(b). Fix $y \in \partial\Omega$. Since the given equation is invariant with respect to rotations and translations in $\mathbb{R}^n$, we can assume $y = 0 \in \partial\Omega$ and $\Omega \subset \mathbb{R}^n_+$. We have

$$\begin{aligned} \pm\Delta\big(u(x) - u(0)\big) &= 0 \ge \Delta v \quad \text{in} \quad \Omega, \\ \pm\big(u(x) - u(0)\big) &\le |g(x) - g(0)| \le |x|^\alpha \le v(x) \quad \text{on} \quad \partial\Omega. \end{aligned}$$

By the comparison principle,

$$|u(x) - u(0)| \le v(x) \le K\,|x|^\alpha, \quad \text{where} \quad K := 1 + c = \frac{n+1}{1-\alpha} > 0.$$

(c). This part is completely similar to Problem 2 in Homework 1.

5. Let $u \in C^2(\mathbb{R}^n)$ be such a function that

$$u \ge 0, \quad \Delta u + u^p \le 0 \qquad \text{on} \quad \mathbb{R}^n, \quad \text{where} \quad p = \text{const} > 0. \tag{1}$$

Show that $u \equiv 0$ on $\mathbb{R}^n$ if $n \ge 3,\ p < n/(n-2)$.

Step 1. Show that

$$M_R := \inf_{B_R} u \ge R^{2-n} \cdot M_1 \quad \text{for all} \quad R \ge 1.$$

Here $B_R := \{x \in \mathbb{R}^n :\ |x| < R\}$.

Step 2. Show that

$$u \ge c \cdot R^2 \cdot M_R^p \quad \text{on} \quad \partial B_{2R},$$

where the constant $c = c(n, p, M_1) > 0$ if $M_1 > 0$.

Step 3. Do the rest.

Step 4, for extra credit. Try to extend the previous argument to show that the statement also holds true for $p = n/(n-2)$.

Solution. Since the case $0 < p \le 1$ has already been considered in Problem 4, Homework 2, we can assume $p > 1$. Let u be a function satisfying the assumptions (1), which is not identically zero. Then u must be strictly positive on $\mathbb{R}^n$, because by the strong maximum principle, is $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^n$, then $u \equiv 0$ on $\mathbb{R}^n$.

For fixed constants $R_1 > R_0 > 0$, introduce the function

$$v(x) := \frac{|x|^{2-n} - R_1^{2-n}}{R_0^{2-n} - R_1^{2-n}} \cdot M_{R_0}, \quad \text{where} \quad M_{R_0} := \sup_{B_{R_0}} u = \sup_{\partial B_{R_0}} u > 0.$$

We have

$$\begin{aligned} u(x) &\ge M_{R_0} = v(x) \quad \text{for} \quad |x| = R_0, \\ u(x) &\ge 0 = v(x) \quad \text{for} \quad |x| = R_1, \\ \Delta u(x) &\le 0 = \Delta v(x) \quad \text{for} \quad R_0 < |x| < R_1. \end{aligned}$$

By the comparison principle, $u(x) \geq v(x)$ for $R_0 \leq |x| \leq R_1$, and sending R_1 to $+\infty$, we get

$$u(x) \geq \frac{|x|^{2-n}}{R_0^{2-n}} \cdot M_{R_0} \quad \text{for} \quad |x| \geq R_0.$$

This implies

$$M_R \geq \frac{R^{2-n}}{R_0^{2-n}} \cdot M_{R_0} \quad \text{for} \quad R \geq R_0.$$

In other words, $R^{n-2}M_R$ is a non-decreasing function of $R \in (0, \infty)$. As a byproduct, we have the estimate in Step 1.

Next, since

$$\inf_{B_{3R}} u =: M_{3R} \geq \frac{(3R)^{2-n}}{R^{2-n}} \cdot M_R = 3^{2-n} M_R,$$

we also have

$$\Delta u \leq -u^p \leq -3^{(2-n)p} M_R^p =: -c_0 M_R^p \quad \text{on} \quad B_{3R}.$$

For an arbitrary point $x_0 \in \partial B_{2R}$, the ball $B_R(x_0)$ is contained in $B_{3R} \setminus B_R$. Compare $u(x)$ with the function

$$w(x) := \frac{|x|^{2-n}}{R^{2-n}} \cdot M_R + \frac{c_0}{2n} \cdot M_R^p \cdot (R^2 - |x - x_0|^2).$$

We have

$$\begin{array}{rcl} u(x) & \geq & \dfrac{|x|^{2-n}}{R^{2-n}} \cdot M_R = w(x) \quad \text{on} \quad \partial B_R(x_0), \\ \Delta u(x) & \leq & -c_0 M_R^p = \Delta w(x) \quad \text{in} \quad B_R(x_0). \end{array}$$

By the comparison principle,

$$u(x_0) \geq w(x_0) = 2^{2-n} M_R + c R^2 M_R^p, \quad \text{where} \quad c := \frac{c_0}{2n} > 0.$$

Since x_0 is an arbitrary point in ∂B_{2R}, it follows

$$M_{2R} \geq 2^{2-n} M_R + c R^2 M_R^p. \tag{2}$$

In particular, we get the estimate in Step 2.

Further, from $M_R \geq M_{2R} \geq c R^2 M_R^p$ and $p > 1$ it follows

$$M_R \leq \left(c R^2\right)^{-1/(p-1)} =: c_1 R^{-2/(p-1)}.$$

On the other hand, $M_R \geq M_1 R^{2-n}$ for all $R \geq 1$. Then we must have

$$2 - n \leq -\frac{2}{p-1}, \quad \text{i.e.} \quad p \geq \frac{n}{n-2}. \tag{3}$$

In the critical case $p = n/(n-2)$, we can write

$$M_R \geq M_1 R^{2-n} = M_1 R^{-2/(p-1)}, \quad R^2 M_R^{p-1} \geq M_1^{p-1} > 0 \quad \text{for all} \quad R \geq 1.$$

Then by virtue of (2),

$$M_{2R} \geq \big(2^{2-n} + cR^2 M_R^{p-1}\big) M_R \geq \big(2^{2-n} + cM_1^{p-1}\big) M_R =: 2^{2-n+\alpha} M_R,$$

where $\alpha = \text{const} > 0$. By iterating this inequality, we obtain

$$M_{2^k} \geq 2^{2-n+\alpha} M_{2^{k-1}} \geq \ldots \geq 2^{k(2-n+\alpha)} M_1 \quad \text{for} \quad k = 1, 2, 3, \ldots.$$

This means that instead of $M_R \geq M_1 R^{2-n}$ we have a stronger estimate

$$M_R \geq M_1 R^{2-n+\alpha} \quad \text{for} \quad R = 2^k,\ k = 0, 1, 2, \ldots.$$

Then instead of (3) we must have

$$2 - n + \alpha \leq -\frac{2}{p-1} = 2 - n.$$

This contradiction proves that the case $p = n/(n-2)$ is also impossible.

Remark. If $p > n/(n-2)$, then there are strictly positive solutions to the above problem. Indeed, for positive constants c and γ, the function $u_0(x) := c|x|^{-\gamma}$ satisfies

$$\Delta u_0(x) = -c\gamma(n-2-\gamma)\,|x|^{-\gamma-2} \quad \text{for} \quad x \neq 0.$$

The right side coincides with $-u_0^p$, if we choose $\gamma > 0$ from the equality

$$\gamma + 2 = p\gamma, \quad \text{i.e} \quad \gamma := \frac{2}{p-1} < n-2,$$

and then c from the equality

$$c\gamma(n-2-\gamma) = c^p, \quad \text{i.e} \quad c := [\gamma(n-2-\gamma)]^{1/(p-1)} > 0.$$

Therefore, the function u_0 satisfies $u_0 > 0$ and $\Delta u_0 + u_0^p = 0$ on $\mathbb{R}^n \setminus \{0\}$.

Further, fix a function η satisfying the properties

$$\eta \in C_0^\infty(\mathbb{R}^n), \quad \eta \geq 0 \quad \text{in} \quad \mathbb{R}^n, \quad \text{and} \quad \int\limits_{\mathbb{R}^n} \eta(x)\,dx = 1.$$

Then the function

$$u(x) := (u_0 * \eta)(x) = \int\limits_{\mathbb{R}^n} u_0(x-y)\eta(y)\,dy = \int\limits_{\mathbb{R}^n} u_0(y)\eta(x-y)\,dy \tag{4}$$

is non-negative and belongs to $C^\infty(\mathbb{R}^n)$. Moreover,

$$|D_{ij}u_0(x)| \leq \text{const} \cdot |x|^{-\gamma-2} \in L^1_{loc}(\mathbb{R}^n), \quad \text{because} \quad \gamma + 2 < n.$$

Hence we can differentiate twice inside of the first integral in (4):

$$\Delta u(x) = \int\limits_{\mathbb{R}^n} \Delta_x u_0(x-y) \cdot \eta(y)\,dy = \big(\Delta u_0 * \eta\big)(x) = -\big(u_0^p\big) * \eta(x).$$

Moreover, by Hölder's inequality,

$$u(x) = \int\limits_{\mathbb{R}^n} u_0(x-y)\eta(y)\,dy \le \left[\int\limits_{\mathbb{R}^n} u_0^p(x-y)\eta(y)\,dy\right]^{1/p} = \left[\big(u_0^p\big) * \eta\right]^{1/p}, \tag{5}$$

which implies

$$\Delta u = -\big(u_0^p\big) * \eta \le -u^p \quad \text{in} \quad \mathbb{R}^n.$$

Thus $u(x)$ is a strictly positive smooth function satisfying (1) with $p > n/(n-2)$.

Alternatively, instead of Hölder's inequality in (5), one can use Jensen's inequality

$$\Phi(u_0 * \eta) \le \big(\Phi(u_0)\big) * \eta, \tag{6}$$

which is true for any convex function Φ (in our case, $\Phi(t) = t_+^p,\ p > 1$). For the proof of (6), note that any convex function is a supremum of linear functions:

$$\Phi(t) = \sup_k (a_k t + b_k), \qquad a_k \quad \text{and} \quad b_k \quad \text{are constants.}$$

For each k,

$$a_k(u_0 * \eta) + b_k = (a_k u_0 + b_k) * \eta \le \big(\Phi(u_0)\big) * \eta,$$

so that

$$\Phi(u_0 * \eta) = \sup_k \left[a_k(u_0 * \eta) + b_k\right] \le \big(\Phi(u_0)\big) * \eta$$

References. The statement of Problem 5, in a more general setting, is contained in the paper [1]. In the case of equality, $u \ge 0,\ \Delta u + u^p = 0$ in $\mathbb{R}^n$, the critical value of p is different. In 1981, Gidas and Spruck [2] proved that for $n > 2,\ 0 < p < (n+2)/(n-2)$, this problem can only have a trivial solution $u \equiv 0$. On the other hand, Pokhozhaev [3] showed that if $p \ge (n+2)/(n-2)$, then there are infinitely many solutions to this problem.

[1] BIDAUT-VERON, M.-F. & POHOZAEV, S., Nonexistence results and estimates for some nonlinear elliptic problems. J. Anal. Math., 84 (2001), 1-49.

[2] GIDAS, B. & SPRUCK, J., Global and local behavior of positive solutions of nonlinear elliptic equations. Comm. Pure Appl. Math., 34 (1981), 525-598.

[3] POKHOZAEV, S.I., On elliptic problems in $\mathbb{R}^n$ with supercritical exponent of nonlinearity.