Math 8583: Theory of Partial Differential Equations: Fall 2007
In-class Midterm Exam, November 7, 2007, 10:10 am - 11:00 am, VinH 211

5 problems, 10 points each, total points: 50.

Books, notes, and calculators are permitted. Please show all your work and make your reasoning clear for each answer.

1. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be a functions satisfying

$$v > 0, \quad \Delta v + \lambda v = 0 \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega,$$

where $\lambda =$const. Show that the problem

$$\Delta u + \lambda u = 1 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

cannot have solutions in $C^2(\Omega) \cap C(\overline{\Omega})$.

2. For a fixed constant $a \in (0,1)$, find

$$\inf_{\mathcal{A}} \int_a^1 r\big[f'(r)\big]^2 dr, \quad \text{where} \quad \mathcal{A} := \Big\{ f \in C^1([a,1]),\ f(a) = 1,\ f(1) = 0 \Big\}.$$

Hint. Note that the infimum of the functional

$$E(u) := \frac{1}{2} \int_{\{a<|x|<1\}} |Du(x)|^2 dx$$

over the set

$$S := \{u \in C^2(\overline{B_1} \setminus B_a) : \qquad u = 1 \quad \text{on} \quad \partial B_a, \quad u = 0 \quad \text{on} \quad \partial B_1$$

is attained on the solution of the problem

$$\Delta u = 0 \quad \text{in} \quad B_1 \setminus \overline{B_a}, \qquad u = 1 \quad \text{on} \quad \partial B_a, \quad u = 0 \quad \text{on} \quad \partial B_1.$$

3. Describe all the vectors $\boldsymbol{v} = (\alpha, \beta) \in \mathbb{R}^2$ such that the function

$$u(x_1, x_2, x_3, x_4) := \Big[x_1^2 + x_2^2 + (\alpha x_3 + \beta x_4)^2\Big]^{-1/2}$$

is harmonic for $x_1^2 + x_2^2 > 0$.

4 (a). (4 points) Let $\alpha \in (0,1)$ be a fixed constant. Show that there is a constant $c = c(n, \alpha) > 0$ such that the function

$$v(x) := |x|^\alpha + c \cdot x_n^\alpha \quad \text{satisfies} \quad \Delta v \le 0 \quad \text{in} \quad \mathbb{R}^n_+ := \{x = (x_1, \dots, x_{n-1}, x_n) :\ x_n > 0\}.$$

4 (b). (3 points) Let Ω be a bounded open convex set in $\mathbb{R}^n$, and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical solution to the problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial\Omega,$$

where g satisfies

$$|g(x) - g(y)| \le |x - y|^\alpha \quad \text{for all} \quad x, y \in \mathbb{R}^n, \quad \text{with} \quad \alpha = \text{const} \in (0,1).$$

Show that there is a constant $K > 0$ such that

$$|u(x) - u(y)| \le K \cdot |x - y|^\alpha \quad \text{for all} \quad x \in \Omega,\ y \in \partial\Omega.$$

4 (c). (3 points) Show that the previous estimate holds true for all $x, y \in \Omega$.

5. Let $u \in C^2(\mathbb{R}^n)$ be such a function that

$$u \geq 0, \quad \Delta u + u^p \leq 0 \qquad \text{on} \quad \mathbb{R}^n, \quad \text{where} \quad p = \text{const} > 0.$$

Show that $u \equiv 0$ on $\mathbb{R}^n$ if $n \geq 3,\ p < n/(n-2)$.

Step 1 (3 points) Show that

$$M_R := \inf_{B_R} u \geq R^{2-n} \cdot M_1 \quad \text{for all} \quad R \geq 1.$$

Here $B_R := \{x \in \mathbb{R}^n : \ |x| < R\}$.

Step 2 (4 points) Show that

$$u \geq c \cdot R^2 \cdot M_R^p \quad \text{on} \quad \partial B_{2R},$$

where the constant $c = c(n, p, M_1) > 0$ if $M_1 > 0$.

Step 3 (3 points) Do the rest.

Step 4, for extra credit up to 5 point. Try to extend the previous argument to show that the statement also holds true for $p = n/(n-2)$.

Remark. For $p > n/(n-2)$, there are strictly positive solutions to the above problem.